The “Rocking Horse Model does Rock”: solving Zambelli’s Puzzle

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Abstract
In 1991, 1992, 2004 and 2007, Stefano Zambelli published a numerical analysis of one of the models developed by Frisch in his seminal book chapter “Propagation problems and impulse problems in dynamic economics.” In the conclusion of his work, he claimed that Frisch’s “rocking horse model is not rocking”! Since then, several authors have taken up his statement. Zambelli’s work raises a sensitive and important question in business-cycle theory: the assessment of Frisch’s 1933 publication. The latter remains difficult to fully understand on several aspects and many authors have tried to clarify the contribution of this publication. However, the present work shows that Zambelli’s conclusion is based on several errors inconsistent with Frisch’s analysis and recommandations. Based on a formal analysis of Zambelli’s work, we demonstrate that instead of clarifying Frisch’s model, Zambelli modifies it, which leads to his erroneous conclusion. By following strictly Frisch’s recommendations, we establish that, contrary to Zambelli’s claim, Frisch’s model does rock.

Keywords: Ragnar Frisch; Rocking horse model; PPIP; Zambelli
JEL : B22, B23, B3

1. Introduction

In 1933, Ragnar Frisch (1895-1973) published his seminal book chapter "Propagation problems and impulse problems in dynamic economics" (the "PPIP") in which he proposed a macrodynamic system to model business cycles. The PPIP's model has been renamed "Frisch's 'rocking horse' model" by some authors,[1] a metaphor used by Wicksell as mentioned by Frisch (1933: 198).[2] His goal was to be able to reproduce the business cycles that were observed at his time: "*The primary cycle of 8.57 years [that] corresponds nearly exactly to the well-known long business cycle"* and "*the secondary cycle [of] 3.50 years, which corresponds nearly exactly to the short business cycle"* (1933: 188-189). Frisch's 1933 publication and its importance in the history of economics have been widely studied, particularly for its contribution to the history of econometrics (Christ 1983; Morgan 1990; Le Gall 1994; Bjerkholt 2007; Louçã 2007), to calibration analysis (Hansen and Heckman 1996), to business cycle analysis (Blatt 1980; Velupillai 1992; Venkatachalam and Velupillai 2012), and for its influence on real business cycle and dynamic stochastic general equilibrium (Stadler 1994; Christiano, Eichenbaum, and Trabandt 2018). This article also contributed to Frisch receiving the first "Bank of Sweden Prize in Economics in Memory of Alfred Nobel" thirty-six years later.

Despite these studies, Frisch's 1933 book chapter remains difficult to fully understand on several aspects. In order to clarify the contribution of Frisch's 1933 book chapter, four economists have provided numerical simulations of this publication. Björn Thalberg (1990: 114) claimed that Frisch's PPIP model cannot fluctuate clearly with the original parameters and conditions. Following Thalberg's analysis, in 1992, Stefano Zambelli published a numerical analysis of Frisch's 1933 book chapter which led him to claim that the "rocking horse is not rocking" (Zambelli 1992: 52). Then, in 2007, this author published an article in the journal *History Of Political Economy* in which he discussed his results from the perspective of the history of economics. In this article, he defended the same conclusion: the "rocking horse model is not rocking" (Zambelli 2007: 147). To support his conclusion, Zambelli had argued that the "*Propagation Problems and Impulse Problems* contains a major error. Frisch major aim was to construct a propagation mechanism that would allow for free oscillations … The problem is that when searching for a complete (either analytical or numerical) solution, which Frisch did not, one discovers that the system is not an oscillating one" (Zambelli 2004: 2-3). Since then, several authors have taken up this conclusion with more or less precautions (Velupillai 1998; Louçã 2001; Bjerkholt 2007; Bjerkholt and Dupont 2010; Dupont-Kieffer 2012; Kolsrud and Nymoen 2014; Boumans 2020). For instance, referring to Zambelli (2007), Bjerkholt and Dupont (2010: 53, n 25) explained that "ironically, Frisch erred in his presentation. The model, which has been studied more than any other business cycle model, did not generate cycles." In 2009, Ricardo Duque, a student of Bjerkholt, provided a mathematical and numerical analysis of Frisch's model for challenging Zambelli's analysis. He argues that Zambelli's criticism can be solved when we consider Frisch's impulse mechanism in the simulation. He conceded that "Zambelli then, through meticulous calculation, rightly pointed out an important flaw in Frisch's model regarding its oscillatory nature" (2009: 37). While Duque claimed that "Zambelli's assertions that PPIP is overall a non-oscillatory model across the board

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[1] See for instance Morgan (1990), Thalberg (1990), Zambelli (1992, 2004, 2007), or Duque (2009).

[2] While Frisch claimed that he borrowed the term "rocking horse" from Wicksell, he confessed in a letter that he did not read Wicksell's publication he cited and that he got the reference to the "rocking horse" from an oral conversation with Åkerman (Boianovsky and Trautwein 2007: 497, n 9). As documented in Boianovsky and Trautwein (2007), the rocking horse metaphor came from Wicksell (1918). See also Dupont-Kieffer (2003: 103).

are wrong" (2009: 1), his demonstration suffers from several problems that weaken his conclusion and demonstration.[3] In 2020, Vincent Carret (2020) provided a computational analysis that extends Frisch's model by applying the Laplace transformation and its inverse. Therefore, contrary to Zambelli, Carret could claim that Frisch's model rocks. The approach proposed in this present work is different from the others, since our work consists in following Frisch's original contribution. It takes advantage of the first comprehensive mathematical analysis of Frisch's 1933 book chapter recently published by Ginoux and Jovanovic (2022a), which has clarified Frisch's analysis and the workability of his model.

While Frisch's book chapter has been recognized for its originality, Zambelli's result shows that Frisch's contribution is unclear. Zambelli's work raises a sensitive and important question in business-cycle theory: the assessment of Frisch's 1933 article. Can we trust Frisch's demonstration? How valid is his article? What is Frisch's contribution if the demonstration does not hold? Furthermore, does Zambelli's analysis make Frisch's model more understandable? Zambelli's work forces us to return to Frisch's publication for investigating all its complexity and assessing its contribution to economic theory.

The present article compares what Frisch did with what Zambelli has undertaken, restricting the comparison to basic assumptions and interpretations. It will demonstrate that Zambelli (1991, 1992, 2004, 2007) is not faithful to the essence of Frisch's model. It will be established that Zambelli's numerical analysis modifies Frisch's propagation model, thus leading him to test a model which is not compatible with what Frisch accomplished. In so doing, it will be established that, contrary to Zambelli's claim, Frisch's propagation model is rocking. First, we briefly present Zambelli's main arguments. We then prove that contrary to Zambelli's claims, and what is commonly accepted in the literature, Frisch did not develop one model in his 1933 book chapter but four distinct models. We show that Zambelli introduced several interpretations that are inconsistent with Frisch's aims and results. We contend that Zambelli's demonstration and numerical investigations do not respect Frisch's recommendations. Finally, we provide a numerical simulation of Frisch's propagation model that respects his recommendations and shows that Frisch's model does rock.

2. Zambelli's Analysis of Frisch's Article

We consider Zambelli's four publications that address his criticism of Frisch's 1933 "rocking horse" model:

1) his working paper published in 1991 by the UCLA department of economics;
2) this first working paper became a chapter published in 1992 in the *Proceedings of the Arne Ryde symposium on Quantitative Methods in the Stabilization of Macrodynamic Systems* edited by his Ph.D. supervisor, Kumaraswamy Velupillai;
3) his second working paper published by the Centre for Comparative Welfare Studies in 2004;
4) this second working paper became an article published in 2007 in the journal *History Of Political Economy.*

[3] In our opinion, the most troublesome problem is that his analysis, and consequently his results, is based on a mixture of Zambelli's analysis and those of Frisch. For instance, his fig. 8, 9, 10, page 30-31 repeats one of Zambelli's interpretations.

In his 1991 and 1992 publications, Zambelli performed numerical investigations to support his criticism of Frisch's "rocking horse" model. Then, in his 2004 and 2007 publications, he put his "discovery that the propagation mechanism of Frisch's … is not a cyclical" (Zambelli 2007: 147) in a historical perspective by considering the debates Frisch had with John M. Clark, Michał Kalecki and Joseph A. Schumpeter "in the thirties about the best way of modeling cyclical evolutions" (Zambelli 2007: 147). In these last two publications, Zambelli summarized his numerical investigations in an appendix that "explains the method used for the computation of the dynamical systems". Specifically, the appendix contains two systems which "can be approximated through numerical integration or *step-by-step* computation" and are those exposed by Zambelli in his 1991 and 1992 publications to support his criticism. Consequently, in his two last publications, he repeated the same argument:

> "In "Propagation Problems" Frisch […] does not show the general solution, that is, the solution in which all the harmonics are summed up. In summing harmonics it is well known that the result of the summation might not be cyclical at all … In other words, Frisch did not show or prove what he said he did" (Zambelli 2007: 152-3).

To evaluate Zambelli's criticism, which is based on his computer simulations, we must focus on his two first publications that detailed his method.

In his work, Zambelli starts by presenting the third and fourth sections of Frisch's publication. He then exposes the results of numerical integrations for reconsidering the propagation problem investigated by Frisch in the fourth section. Zambelli explains that he "will concentrate only upon the propagation problem" (1992: 28), considering "that section 5 [relative to the impulse problem] was only an addendum to the more important previous sections in which he constructed the (cyclical) propagation mechanism" (2007: 158). He adds that he has "worked out and reproduced (exactly) Frisch's numerical examples" (1992: 27). "With the aid of a computer [he has] simulated Frisch's examples, recomputed the roots and reproduced the same values and graphs that Frisch presented in his original paper" (1992: 33). He also uses the term "exactly" several times in each of his four papers for emphasizing that his work would be perfectly consistent with Frisch's. Zambelli acknowledges that Frisch's model, used for solving the propagation problem, oscillates but its "oscillations do take place by construction" (1992: 33), and the cycles generated are not realistic due to the initial conditions of the model. Indeed, "the past history consistent with the assumed initial conditions is given by an oscillation that is highly unlikely, to say the least" (1992: 35). For this reason, Zambelli argues that "Frisch's own argument is rather *incomplete*" (1992: 37). Therefore, he "completes" the model developed for solving the propagation problem by interpreting what Frisch should have done if he had remained consistent with his "simplified systems without oscillations":

> "Consistent with the argument previously developed [Frisch] should have confronted the qualitative behaviour of SYSTEM 1 [namely the model with oscillations] with the qualitative behaviour of equation (6) [from the model without oscillations]. In other words, the single members appearing on the right-hand side of the general solution of SYSTEM 1 (equation (11)) should have been summed so as to give the complete solution" (1992: 37).

Having completed Frisch's model, Zambelli provides numerical simulations and shows that *cycles* do not exist with his completed model:

> "The main result of the paper is that the model presented in PPIP, the so-called *propagation mechanism*, is not intrinsically cyclical. Therefore it is not 'a macro-dynamic model giving rise to oscillations,' as claimed by Frisch. When the system is perturbed from the equilibrium, i.e., is subject to an external shock, it evolves back to the equilibrium position in a non-cyclical manner. Using Frisch own metaphor, we have the paradoxical result that the rocking horse model does not *rock*" (1992: 52).[4]

Contrary to what claims Zambelli, we will establish in the next sections that the Rocking horse does rock.

3. Frisch's Aim and Models

As mentioned in the previous section, Zambelli explains that he only deals with the propagation problem (PP). However, as he clarifies, he "completes" Frisch's PP model (or "system" if we keep Frisch's terminology)[5] by borrowing elements from two other different models developed by Frisch (namely the "system without oscillations" and the "macrodynamic system giving rise to oscillations" thanks to exogenous impulses). In this case, the question is whether it is consistent with Frisch's demonstration to borrow elements from the model without oscillations and the model with impulsions for completing the propagation problem, as Zambelli does.

To date, the literature has considered that Frisch's analysis in his book chapter like a solo model that is progressively complexified.[6] Such interpretation is consistent with some of Frisch's sentences; however, is not fully consistent with the mathematical hypotheses and formulations proposed by Frisch. Specifically, as explained in Ginoux and Jovanovic (2022a), in his essay, Frisch studied successively the different aspects of the propagation and impulse problems in business cycles. He proposed several models, and each of them was developed with a specific goal. Even if some connections between these models exist, Frisch never presented them as one general model that would have allowed the borrowing that Zambelli makes. The way Frisch organized his article reflects this.

In section 3 (pp. 175-181), called "Simplified systems without oscillations" (note the plural), Frisch developed a mathematical model without oscillations (let's call it MODEL 1). His goal was to study the relations between the macroeconomic variables of the Tableau Economique (Frisch 1933: 174) from which he can determinate the trend of business cycles. Then Frisch explained that the mathematical MODEL 1 needs to be generalized thanks to a theoretical explanation of the fluctuations in order to be compatible with the observed oscillations. So doing,

[4] See also Zambelli (2004: 24; 2007: 163), and Duque (2009).

[5] At Frisch's time, the term "system" referred to a set of ordinary differential equations while the term "model" referred to a mathematical representation of a natural or artificial phenomenon (Hirsch and Smale 1974). Because Frisch developed a set of ordinary differential equations, he used the term "system." Nowadays, "model" is commonly used therefore we are using this term in the present article.

[6] Frisch's essay was published in the book *Economic Essays in Honor of Gustav Cassel* for which contributors were invited to write a chapter. In other words, Frisch did not have to follow the selection process as we have in a scientific journal, and he did not have to format his article according to the standards of a scientific article. Moreover, Frisch sent his chapter with a delay, and consequently it could not be reviewed by the editors as originally planned (Bjerkholt 2007: 473-474).

he discussed literally four economic "models" at the end of his third section. Among these four economic models, Frisch considered "Aftalion's point of view with regard to production" as the most relevant one.

However, Frisch explained that Aftalion's idea cannot be used such as he formulated because "he does not have as many equations as unknowns" (1933: 181). Consequently, he dedicated his section 4 (pp. 181-197), called "A macro-dynamic system giving rise to oscillations" (note the singular), to "the discussion of such a system." He thus developed a mathematical model of damped oscillations with delay (let's call this propagation model MODEL 2). This model introduced a propagation mechanism that allowed Frisch to account for the economic oscillations. Let's us remind that the MODEL 2 is a system composed of the three following equations (Frisch 1933, p. 177 & 182, eq. 2, 3.3, 4).

$$\begin{cases} \dot{x} = c - \lambda(rx + sz) \\ y = mx + \mu\dot{x} \\ \varepsilon\dot{z} = y(t) - y(t-\varepsilon) \end{cases} \tag{1}$$

Where the variables are y, the annual investment (called "the yearly production of capital goods"), x, the annual consumption (called "the yearly production of consumer goods"), and z the carry-on activity in capital goods (i.e., the lag "between capital goods ordered and capital goods delivered" (Frisch 1931: 652)). We have six parameters: m is the depreciation of capital, μ expresses the size of capital stock that is needed directly and indirectly in order to produce one unit of consumption per year, c expresses a tendency to maintain and perhaps expand consumption λ expresses the reining-in effect of the encaisse désirée, r expresses consumption habits, and s is the nature of existing monetary institutions; and a constant ε, which is the number of units of time needed between the investment and its maturity. This model generates oscillations which are damped by the "encaisse désirée." In this model, the damped oscillations disappeared after a damped period.[7] It is precisely this model that Zambelli (1991, 1992, 2004, 2007) discusses. Having obtained his model, Frisch calibrated it with a numerical step-by-step solution until the values of parameters fit with the observations (that are a primary cycle of 8.57 years, a secondary cycle of 3.50 years, and a tertiary cycle of 2.20 years). Frisch (1933: 185-188) obtained the following values: $c = 0.165$, $\lambda = 0.05$, $r = 2$, $s = 1$, $m = 0.5$, $\mu = 10$, and $\varepsilon = 6$.

Having solved the propagation problem, Frisch discussed the impulse problem which was investigating in the last two sections.

In section 5 (pp. 197-203), called "Erratic shocks as a source of energy in maintaining oscillations," Frisch solved the impulse problems by introducing exogenous and erratic impulses that can maintain oscillations in a model of free damped oscillations (let's call this propagation and impulse model MODEL 3). For doing this, he used some results from Eugen Slutsky (1927), George Udny Yule (1927), and Harold Hotelling (1927) who had demonstrated that irregular

[7] As explained by Tenenbaum and Pollard (1963: 351), "a *damped period* is the time it takes, starting at the equilibrium position, to make one complete oscillation". As clarified by Andronov *et. al.* (1937 [1949]: 16), it is "an interval of time between two successive passages of the system through the position of the equilibrium (in the same direction) or between two successive maximum deviations (on one on the same side)".

damped oscillations may be transformed into regular oscillations similar to ones we observe in business cycles. The introduction of these exogenous random shocks should have given birth to the famous "propagation-impulse model." However, Frisch never provided an explicit mathematical formulation for his "propagation-impulse model," neither in 1933 nor afterwards. In fact, and as Frisch mentioned (1933: 199), he only provided the main characteristics of his MODEL 3.[8]

In the sixth and last section (pp. 203-205), called "The innovations as a factor in maintaining oscillations," Frisch discussed the economic framework for another "propagation-impulse model" based on Schumpeter's theory of innovations (let's call this propagation and impulse model MODEL 4). To support his argument, Frisch provided a mechanical analogy of a pendulum and referred to "auto-maintained oscillations."[9] In this perspective, the innovations are endogenous impulses which could maintain the oscillations. It is worth noting that Frisch never provided any mathematical formulation of his MODEL 4.

As we can see, Frisch discussed four major models in his book chapter in order to "get systems where the theoretical movement will contain oscillations" (Frisch 1933: 175); each of them was developed with a specific goal. In addition, these models do not use the same kind of oscillations (free damped oscillations, maintained oscillations by erratic shocks, self-maintained oscillations) as the next section will show.

4. Zambelli's Interpretations and Terminologies are inconsistent with Frisch's Demonstration

Combining elements from three different models, as Zambelli does for "completing" Frisch's MODEL 2, must be performed in a rigorous framework. Unfortunately, when he completes Frisch's model Zambelli introduces several interpretations inconsistent with Frisch's demonstration. Here are three examples regarding the oscillations.

First, as previously mentioned, Zambelli argues that "Frisch major aim was to construct a propagation mechanism that would allow for free oscillations" (2004: 2) and "consistent with the argument previously developed he [Frisch] should have … summed [the single members appearing on the right-hand side of the general solution of SYSTEM 1 (equation (11))] to give the complete solution. In fact, we know that the sum of harmonic (or trigonometric) functions may well give rise to monotonic behaviour, and this is a concrete possibility" (1992: 38).[10] In his last article, Zambelli repeats the same argument:

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[8] Frisch clarified that "For a more detailed mathematical analysis the reader is referred to a paper to appear in one of the early numbers of *Econometrica*" (Frisch 1933: 199). Unfortunately, such a paper has never been published. While Frisch did not provide an explicit mathematical formulation for his "propagation-impulse model," his model is complete in the sense that Frisch found the values of the periods of the economic cycles by calibrating this model. Therefore, when Duque (2009: 48) argues that this model is incomplete, it must be understood that it is mathematically incomplete.

[9] On the way Frisch and other economists investigated relaxation oscillations and self-maintained oscillations for modeling on business cycles in the 1930s, see Ginoux and Jovanovic (2022b).

[10] It is worth mentioning that this sentence is mathematically false. Indeed, harmonic functions may refer either to trigonometric functions, which are periodic, or to other functions, which are not necessarily periodic (Arekat 2020). Consequently, Zambelli's statement does not hold for trigonometric functions; in this case, the sum of trigonometric functions does not give rise to monotonic behavior but obviously to cyclical behavior.

> "In "Propagation Problems" Frisch confronts the reader with an example in which a sample (trend, primary, secondary, and tertiary) of the infinite number of harmonics that make the total solution of a mixed difference differential equation is shown to exhibit oscillating evolutions. **He does not show the general solution, that is, the solution in which all the harmonics are summed up.** In summing harmonics it is well known that the result of the summation might not be cyclical at all" (Zambelli 2007: 152-3, emphases added).

Thus, Zambelli considers that Frisch should have made the sum of harmonic functions, namely the sum of the trend, primary, secondary and tertiary cycles (solutions of Frisch's MODEL 2). First, let's notice that the terminology "sum of harmonics (or trigonometric) functions" used by Zambelli (1992: 38) is confusing and incorrect since these functions are in fact "damped sine curves," as recalled by Frisch (1933: 185) and not only harmonic or trigonometric functions in his meaning.

In fact, the terminology used by Frisch for describing his propagation mechanism is often different from that we currently use.[11] The term "free oscillations" is one of the confusing terms. Frisch started his article by claiming that "THE majority of the economic oscillations … seem to be explained most plausibly as free oscillations. In many cases they seem to be produced by the fact that certain exterior impulses hit the economic mechanism and thereby initiate more or less regular oscillations" (1933: 171). Although Frisch used the terminology "free oscillations," the term "free oscillations" is different from the current meaning. Indeed, by opposition to Frisch's sentence, "free oscillations" are nowadays only associated to "undamped oscillations" or "harmonic oscillations," and all of them have regular oscillations or more exactly *periodic* oscillations[12]. But it is not the case in Frisch's article. In his second paragraph, Frisch clarified his conception by explaining that some "exterior impulses ... initiate more or less regular oscillations" (1933: 171). Given that free oscillations or harmonic oscillations do have regular oscillations, in this sentence Frisch could not refer to what we call nowadays "free oscillations". In the second paragraph, Frisch was more specific:

> "The most important feature of the free oscillations is that the length of the cycles and the tendency towards dampening are determined by the intrinsic structure of the swinging system, while the intensity (the amplitude) of the fluctuations is determined primarily by the exterior impulse. An important consequence of this is that a more or less regular fluctuation may be produced by a cause which operates irregularly."

Considering this explanation, it is clear that for Frisch, "free oscillations" mean "free damped oscillations." Therefore, "free oscillations" must be understood as "free damped oscillations" in the whole article, in particular in Frisch's MODEL 2, and should not be associated with "harmonic oscillations," according to the current terminology. When Zambelli (2004: 2) explains that Frisch constructed a propagation mechanism that would allow for free oscillations, and that he should have summed up all the harmonics, he should have considered that free oscillations mean for Frisch "free damped oscillations." Therefore, Zambelli misinterprets Frisch's reasoning when he claims

[11] Morgan (1990: 89) already pointed out the problem of terminology in Frisch's publications: "Frisch usually presented his work in this field as if it were a brand new approach to a problem and invented his own new terminology". In fact, as with any precursor, the terminology he used and introduced was not fixed yet at his time, and consequently nowadays it is difficult to read Frisch's book chapter correctly without replacing some of his terms.

[12] See Andronov *et. al.* (1937 [1949]: 16).

that Frisch did not sum up the harmonic oscillations in his MODEL 2, because the general solution of his MODEL 2 is a sum of *damped sine curves* and is not a sum of harmonic oscillations (namely sinus or cosine functions) as recalled by Frisch (1933: 183). Moreover, following Zambelli's claim that Frisch should have summed up all the harmonics, his reconstruction is completely inconsistent with Frisch's aim, which was only to establish the periods of these three business cycles and not all possible cycles.

Second, Zambelli makes another error regarding the oscillations. In his article, Frisch considered a second kind of oscillation, the auto-maintained oscillations, when he discussed the mechanical analogy of Schumpeter's idea in his section 6 (the MODEL 4). Zambelli explains that this metaphor "is not as simple or appealing as that of the rocking horse (or pendulum) [namely MODEL 3], which is kept oscillating by erratic shocks, but it [namely MODEL 4] has the same feature [than MODEL 3] of explaining the maintenance of cycles thanks *to an exogenous force* that provides energy to the system" (2007: 159). Contrary to Zambelli's interpretation, Frisch's metaphor, which is based on "auto-maintained oscillations" (1933: 203), explains the maintenance of cycles thanks to an ENDOGENOUS force and not an exogenous one.[13]

Third, in his numerical simulations, Zambelli claims that he uses "the same [numerical values] used by Frisch in the sequel of the article: $\lambda = 0.05$, $r = 1$, $s = 1$, $m = 0.5$, $\mu = 10$, $c = 0.165$" (1991: 4; 1992: 30). As we can observe, the value of the parameter r is not the same as the one used by Frisch, which is equal to 2 in Frisch's book chapter. However, in his 2004 and 2007 publications, Zambelli changes the value of the parameter r and takes the same as Frisch. Unfortunately, Zambelli never provides the program nor the initial conditions for the "past history" he uses to make his numerical investigations, therefore, when we cannot know if the value of the parameter r equal to 1 in Zambellis's 1991 and 1992 publications is just a typo or if the model used by Zambelli to support his criticism differs from that of Frisch.

These three examples show again that Zambelli's analysis of Frisch's models is not consistent with Frisch's original demonstration.

5. Zambelli's Mathematical Errors

As explained, Zambelli aims at testing the general solution of Frisch's MODEL 2 with some numerical simulations. In his "main critical points and remarks," Zambelli claims that "the way in which Frisch constructs the argument, when he shows the 'propagating' evolution of the primary, secondary and tertiary cycles, is rather arbitrary because full information, with respect to the position and the speed of the single components as given at point of time t_o, is assumed" (1992: 38). Zambelli uses this key argument for justifying that if Frisch had used the "full information," his model would not have produced oscillations. He summarizes this point in his 2007 article:

> "A numerical step-by-step solution of Frisch's model shows that the system is not at all oscillating … . **He does not show the general solution, that is, the solution in which all**

[13] As Diner (1992: 340) explained, "Self-oscillators are open nonlinear systems with energy (or matter) input and energy (or matter) dissipation with feedback on the input. It is the dissipation that regulates the feedback. The flow of the energy source is controlled by the state of the system, which is the very principle of the feedback". On Schumpeter's innovations, see for instance Shionoya (1996: 296).

the harmonics are summed up. In summing harmonics it is well known that the result of the summation might not be cyclical at all, a trivial example being represented by two sinusoidal functions having the same amplitude and being out of phase for 180 degrees: the sum of the two harmonics is a constant function, a straight line. What Frisch did was to show the fluctuating behavior of the individual components of the general solution, but he did not sum them up. **If he had done so, he would have discovered that his was not at all a cyclical model but quite the contrary: he would have discovered something very similar to a straight line … .** In other words, Frisch did not construct an oscillating mechanism; indeed, quite the contrary. Hence **Frisch did not show or prove what he said he did**" (Zambelli 2007: 152-3, emphases added).[14]

In other terms, according to Zambelli, Frisch should have summed up the solutions of eq. 23 a, b, c and d of his MODEL 2 (page 192 of Frisch's article – see Fig. 4 reproduced in the appendix). This criticism is not mathematically relevant. Let us remind that eq. 23 a, b, c and d of his MODEL 2 result of a trend (eq. 16 in Frisch) plus the oscillations of the cyclical components, i.e., the three variables x, the annual consumption, y, the annual investment, and z the carry-on activity in capital goods (eq. 18 in Frisch). Mathematically, Zambelli makes several errors.

First of all, let us come back the "trivial example" used by Zambelli to support his demonstration that "it is well known that the result of the summation might not be cyclical at all, a trivial example being represented by two sinusoidal functions having the same amplitude and being out of phase for 180 degrees: the sum of the two harmonics is a constant function, a straight line" (Zambelli 2007: 153). It is true that the sum of "two sinusoidal functions having the same amplitude and being out of phase for 180 degrees" is a straight, but only because they have the same amplitude. If such amplitudes are different and normalized, i.e. if their sum is equal to the unity (take for example 0.1 and 0.9) then, contrary to what Zambelli claims, such a sum exhibits harmonic oscillations or cyclical behaviour in Zambelli's terms.

Zambelli makes another error when he does the calculation of eq. 23 a, b, c and d for the trend and the three cycles identified by Frisch, because he makes an equally weighted sum of the coefficients for each of these components. Thanks to his calculations, Zambelli is able to show the lack of oscillations. Unfortunately, his calculation is erroneous. Frisch made clear that we should not proceed like Zambelli does:

"A given set (18) (for a given j) does not –taken by itself– satisfy the dynamic system consisting of (3.3), (2) and (4) [namely MODEL 2]. **It will do so only if the constant structural $c = 0$. If $c \neq 0$** the constant terms a_*, b_* and c_* must be added to (18) in order to get a correct solution. If these constant terms are added, we get functions that satisfy the dynamic system, and that have **the property that any linear combination of them (with constant coefficients) satisfy the dynamic system provided only that the sum of the coefficients by which they are linearly combined is equal to unity**" (1933: 191, emphases added).

In fact, the series, i.e., the linear combination of the trend, primary, secondary, and tertiary cycles, etc. (namely the sum of eq. 23 a, b, c and d) which represents the general solution of Frisch's

[14] See also Zambelli (1992: 38).

MODEL 2 oscillates provided that the sum of its coefficients is equal to unity as recalled by Frisch above.[15] Concerning the general solution of MODEL 2, when Zambelli decides to verify if it oscillates (rocking) or not, he does not pay attention to the fact that the sum of the coefficients must be equal to unity and must not be equally pondered. Indeed, although he does not provide the information, we can deduce from Zambelli's figure 4.5 (Zambelli 1992: 38) that he uses the following values in his numerical investigations: $a_0 = 1,\ a_1 = 1,\ a_0 = 1,\ a_3 = 1$ (Figure 1).[16]

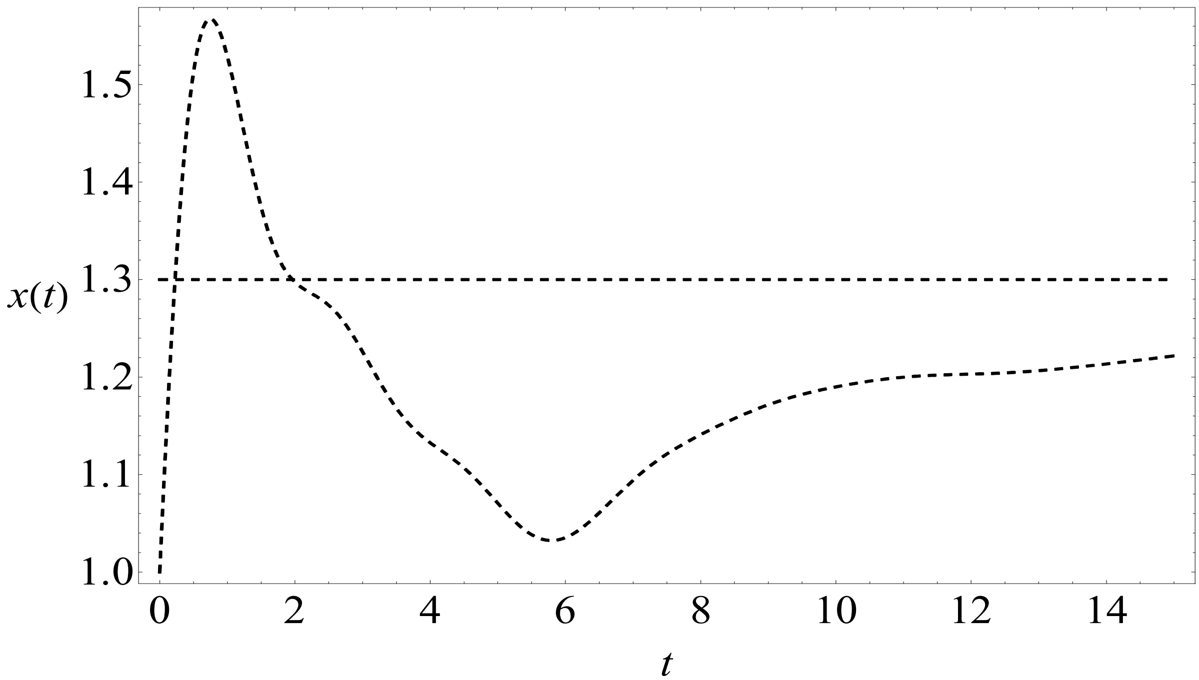

Fig. 1. Reconstruction of Zambelli's (1992: 38) figure 4.5.

So, contrary to Frisch's recommendation, Zambelli does not normalize the solution.[17] This is a serious mathematical error regarding Frisch's demonstration. Indeed, Zambelli makes a linear combination of the solutions with constant coefficients all equal to 1, therefore their sum is necessarily superior to unity. If we do not make a normalization, and if all the coefficients have the same weight, we may effectively obtain no oscillation (like Zambelli does). In other words, from a mathematical perspective, the lack of oscillation obtained by Zambelli is based on a mathematical error.

Contrary to Zambelli, if we follow Frisch's recommendations (namely that the sum of the coefficients must be equal to unity), the MODEL 2 does rock. For instance, figure 2 is a numerical simulation of the general solution of Frisch's MODEL 2 (i.e. $a_0x_0(t)+a_1x_1(t)+a_2x_2(t)+a_3x_3(t)$) with the same value for the parameters set of Frisch (c = 0.165, λ = 0.05, r = 2, s = 1, m = 0.5, μ = 10, ε = 6) and the following coefficients: $a_0 = 0.01,\ a_1 = 0.1,\ a_0 = 0.2,\ a_3 = 0.699$.[18] We observe

[15] What Frisch done by hand is the computation of Lyapunov characteristics exponents which is nowadays performed with the so-called Gram Schmidt reorthonormalization (GSR) procedure on the vector frame. See Wolf et al. (1985).
[16] All programs used by the authors and performed in Mathematica are available by simple request to the authors.
[17] It is worth mentioning that Duque (2009: 30-31) repeated Zambelli's misinterpretation.
[18] Frisch did not provide any value in his article for these coefficients since he did not intend to make such a sum.

that the solution—the linear combination of the trend, primary, secondary, and tertiary cycles—exhibits free damped oscillations.

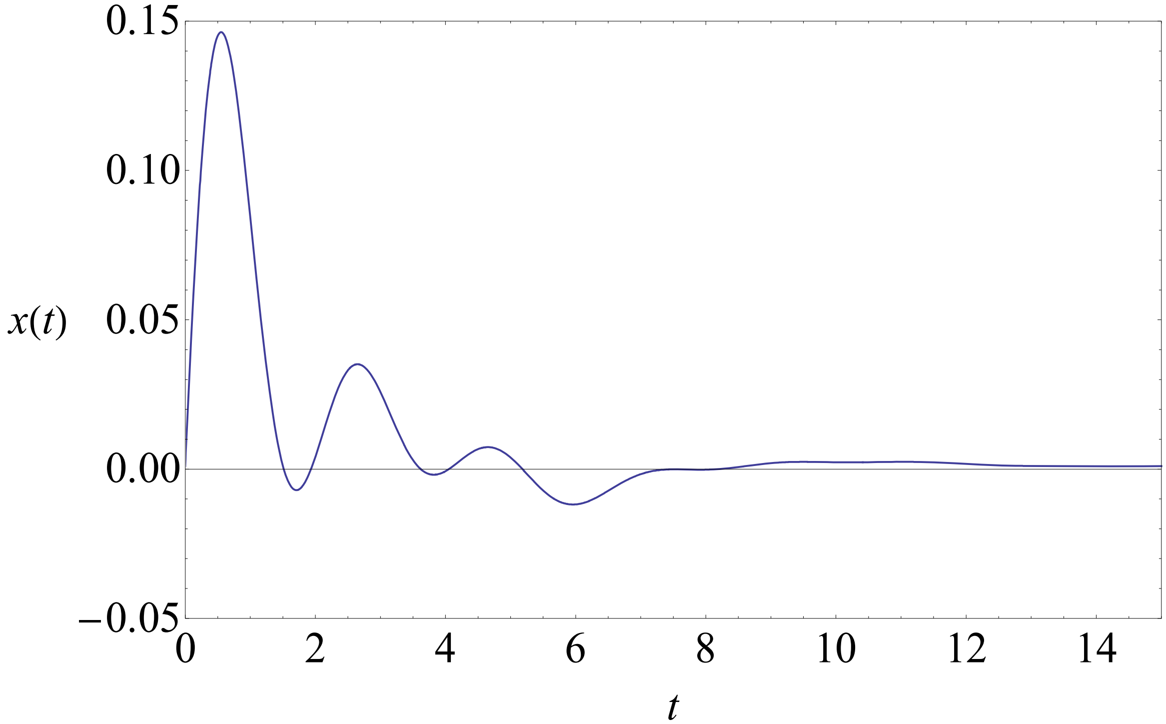

Fig. 2. Numerical simulation of the general solution of Frisch's MODEL 2 with coefficients (weights) consistent with Frisch's recommendations.

6. Conclusion

With his four publications, Zambelli aims at clarifying the contribution of Frisch's 1933 book chapter, which is still remained difficult to fully understand. He makes a reconstruction of Frisch's "macro-dynamic system" by summing the *damped sine curves*, solutions of such a propagation model (MODEL 2) to show that the result does not oscillate. Since then, several authors have taken up this conclusion, clamming that Frisch's demonstration does not hold. In so doing, these authors have raised a sensitive and important question in business-cycle theory: are Frisch's 1933 contribution and results overestimated? This article has shown that Zambelli's interpretation, terminology, and conclusion are not consistent with Frisch's aims and demonstration. It has been shown that Frisch developed four major distinctive models in his 1933 article, and not one as Zambelli and the literature suggest. It has been demonstrated that, despite Frisch's recommendations, Zambelli does not normalize the coefficients involved in the linear combination of the sum of harmonics representing the general solution of Frisch's system. That leads him to a misleading conclusion about the fact that the rocking horse does not rock. Therefore, Zambelli's assertions are not supported by an acceptable demonstration, and his conclusion is inconsistent with Frisch's analysis. Our work suggests that Zambelli's analysis of history of economics of the 1930s needs to be reassessed. Our work also suggests that what Frisch accomplished in his 1933 article as well as his debates with other economists on business cycles in the 1930s still deserve clarification and further research.

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Appendix – Proof that Frisch's MODEL 2 is rocking[19]

According to Boumans (2004: 57), *phase* and *amplitude* ("weight") of the general solution to Frisch's system (1) (namely MODEL 2) is given by an infinite sum of exponentials (A1), which is defined by an initial movement on a certain period.

$$\begin{cases} x = a_* + \sum_k a_k e^{\rho_k t} \\ y = b_* + \sum_k b_k e^{\rho_k t} \\ z = c_* + \sum_k c_k e^{\rho_k t} \end{cases} \qquad \text{(A1, [Frisch 1933, eq. 8, p. 183])}$$

By replacing *k* by *j* and by posing $\rho_j = -\beta_j + i\alpha_j$, Frisch's equations (A1), i.e., the solution of his MODEL 2, can be rewritten as follows:

$$\begin{cases} x(t) = a_* + \sum_j a_j e^{\rho_j t} = a_* + \sum_j a_j x_j(t) \\ y(t) = b_* + \sum_j b_j e^{\rho_j t} = b_* + \sum_j b_j y_j(t) \\ z(t) = c_* + \sum_j c_j e^{\rho_j t} = c_* + \sum_j c_j z_j(t) \end{cases} \qquad \text{(A2)}$$

where the "cyclical components" read:

$$\begin{cases} x_j(t) = A_j e^{-\beta_j t} \sin(\phi_j + \alpha_j t) \\ y_j(t) = B_j e^{-\beta_j t} \sin(\psi_j + \alpha_j t) \\ z_j(t) = C_j e^{-\beta_j t} \sin(\theta_j + \alpha_j t) \end{cases} \qquad \text{(A3, [Frisch 1933, eq. 18, p. 190])}$$

Thus, Frisch computed the phase and amplitude of the *trend* and the three first *cyclical components* of the solution (A3) of his model (1), i.e. $j = 0, 1, 2$ and 3. In order to simplify and also to reduce the length of our demonstration, we will only focus on the primary cycle of (A3), i.e. $j = 1$. Frisch (1933: 192) explained that the primary cycle, solution of his system (1), is represented by the functions:

$$\begin{cases} x_1(t) = 0.6816 e^{-\beta_1 t} \sin(\alpha_1 t) \\ y_1(t) = 5.4585 e^{-\beta_1 t} \sin(1.9837 + \alpha_1 t) \\ z_1(t) = -10.662 e^{-\beta_1 t} \sin(1.9251 + \alpha_1 t) \end{cases} \qquad \text{(A4, [Frisch 1933, eq. 23b, p. 192])}$$

Starting from the first equation of (A3) and by considering that $j = 1$ (*primary cycle*), we have:

$$x_1(t) = A_1 e^{-\beta_1 t} \sin(\varphi_1 + \alpha_1 t)$$

But according to Frisch (1933: 187, Tab. I), $\varphi_1 = 0$ and $A_1 = 1/2\alpha_1$. Moreover, Frisch (1933: 187, Tab. I) computed the values of α_1 and β_1 which have been provided in Fig. 3.[20]

[19] For a complete mathematical analysis of Frisch's 1933 book chapter, see Ginoux and Jovanovic (2022a).

[20] In Ginoux and Jovanovic (2022a), we made Frisch's calculations more accurate.

TABLE I

CHARACTERISTIC COEFFICIENTS OF THE COMPONENTS OBTAINED

	Trend ($j = 0$)	Primary Cycle ($j = 1$)	Secondary Cycle ($j = 2$)	Tertiary Cycle ($j = 3$)
Frequency .. .. α	$\rho_0 = -0{\cdot}08045$	0·73355	1·79775	2·8533
Period .. .. $p = \frac{2\pi}{\alpha}$		8·5654	3·4950	2·2021
Damping exponent .. .. .. β		0·371335	0·5157	0·59105
Damping factor per period .. .. $e^{-2\pi\beta/\alpha}$		0·0416	0·1649	0·2721

Fig. 3. Characteristic coefficients of the components obtained from Frisch (1933: 187).

So, with $\alpha_1 = 0.733552$, we have $A_1 = 0.6816$ and then $x_1(t) = 0.6816e^{-\beta_1 t}\sin(\alpha_1 t)$. By using the second equation of the system (1), i.e., $y = mx + \mu\dot{x}$, we obtain that:

$$y_1 = mx_1 + \mu\dot{x}_1 = 0.5\left[0.6816e^{-\beta_1 t}\sin(\alpha_1 t)\right] + 10\left[-0.2531e^{-\beta_1 t}\sin(\alpha_1 t) + 0.4999e^{-\beta_1 t}\cos(\alpha_1 t)\right]$$

since $\dot{x}_1(t) = -0.2531e^{-\beta_1 t}\sin(\alpha_1 t) + 0.4999e^{-\beta_1 t}\cos(\alpha_1 t)$ with $\beta_1 = 0.371335$. This leads to:

$$y_1(t) = \left[-2.1902\sin(\alpha_1 t) + 5\cos(\alpha_1 t)\right]e^{-\beta_1 t}$$

Thus, with the classical method of normalization, we find that $\sqrt{(-2.1902)^2 + 5^2} = 5.4585$. By posing: $\cos\phi = -2.1902/5.4585$ and $\sin\phi = 5/5.4585$, we deduce that $\phi = 1.9837$ (in radians unit). By using the trigonometric formula: $\sin a\cos b + \sin b\cos a = \sin(a+b)$, we have:

$$y_1(t) = 5.4585e^{-\beta_1 t}\sin(1.9837 + \alpha_1 t)$$

So, $y_1(t)$ is indeed a solution of Frisch's system (1). This proof can be extended to all other variables and for all cycles. Thus, we obtained for the variables (x_j, y_j, z_j) where $j = 1, 2, 3$ provide the same results as those obtained by Frisch (1933: 192, eq. 23a, 23b, 23c, 23d) and reproduced in Fig. 4 below.

$$
(23a) \qquad \begin{cases} x_0 = 1\cdot 32 - 0\cdot 32e^{-0\cdot 08045t} \\ y_0 = 0\cdot 66 + 0\cdot 09744e^{-0\cdot 08045t} \\ z_0 = 0\cdot 66 + 0\cdot 12512e^{-0\cdot 08045t} \end{cases}
$$

$$
(23b) \qquad \begin{cases} x_1 = 0\cdot 6816e^{-\beta_1 t} \sin \alpha_1 t \\ y_1 = 5\cdot 4585e^{-\beta_1 t} \sin (1\cdot 9837 + \alpha_1 t) \\ z_1 = -\ 10\cdot 662e^{-\beta_1 t} \sin (1\cdot 9251 + \alpha_1 t) \end{cases}
$$

$$
(23c) \qquad \begin{cases} x_2 = 0\cdot 27813e^{-\beta_2 t} \sin \alpha_2 t \\ y_2 = 5\cdot 1648e^{-\beta_2 t} \sin (1\cdot 8243 + \alpha_2 t) \\ z_2 = -\ 10\cdot 264e^{-\beta_2 t} \sin (1\cdot 7980 + \alpha_2 t) \end{cases}
$$

$$
(23d) \qquad \begin{cases} x_3 = 0\cdot 17524e^{-\beta_3 t} \sin \alpha_3 t \\ y_3 = 5\cdot 0893e^{-\beta_3 t} \sin (1\cdot 7582 + \alpha_3 t) \\ z_3 = -\ 10\cdot 147e^{-\beta_3 t} \sin (1\cdot 7412 + \alpha_3 t) \end{cases}
$$

Fig.4. Frisch's "cyclical components" $x_j(t)$, $y_j(t)$, $z_j(t)$ for $j = 0, 1, 2$ and 3, from Frisch (1933: 192).