Noname manuscript No.
(will be inserted by the editor)

Frisch's propagation-impulse model: A comprehensive mathematical analysis

Jean-Marc Ginoux & Franck Jovanovic

Received: date / Accepted: date

Abstract Frisch's 1933 macroeconomic model for business cycles has been extensively studied. The present study is the first comprehensive mathematical analysis of Frisch's model. It provides a detailed reconstruction of how the model was built. We demonstrate the workability of Frisch's PPIP model without adding hypotheses or changing the value of Frisch's parameters. We prove that 1) the propagation model oscillates; 2) the PPIP model is mathematically incomplete; 3) the latter could have been calibrated by Frisch; 4) Frisch's analysis and demonstration are based on Poincaré's methodology

Keywords Ragnar Frisch · propagation-impulse model · Rocking horse model · theory of oscillations

1 Introduction

In 1933, Ragnar Frisch (1895-1973) published his seminal book chapter "Propagation problems and impulse problems in dynamic economics" (the "PPIP") in which he proposed a macrodynamic system to model business cycles. This model, which has been renamed "Frisch's 'rocking horse' model" by some authors, is one of the first attempts to explain economic fluctuations through the use of a mathematical model. This book chapter also contributed to Frisch receiving the first "Bank of Sweden Prize in Economics in Memory of Alfred Nobel" thirty-six years later. As Kenneth Arrow reminded, "Ragnar Frisch belongs to that handful of pioneers who have [...] begun a transformation

J.M. Ginoux
Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France.
E-mail: ginoux@univ-tln.fr,
F. Jovanovic
School of Business Administration, Teluq University, Canada,
LEO, Université d'Orléans, France.
E-mail: franck.jovanovic@teluq.ca

of all economics with respect to the standards of scientific inquiry" (1960, p. 175).

"The 1933 essay of Frisch has thus never ceased to be part of the current theoretical discussion" (1960, p. 175). Indeed, it has been widely studied in the literature and its origins deeply investigated, and particularly for the calibration analysis it introduced in econometrics (Backhouse, 2015; Bjerkholt & Dupont, 2010; Blatt, 1980; Boumans, 2005; Christ, 1983; Hansen & Heckman, 1996; Le Gall, 1994; Louçã, 2007; Morgan, 1990; Samuelson, 1974; Velupillai, 1992; Venkatachalam & Velupillai, 2012).

Moreover, this essay also paved the way to the real business cycle (RBC) and the dynamic stochastic general equilibrium (DSGE) methods for analyzing business cycles; both approaches have developed macrodynamic models with exogenous and random shocks and with the use of calibration (Christiano *et al.*, 2018). Despite these investigations, the understanding of this model remains extremely difficult and partly obscured. First of all, the reading of Frisch's book chapter introduced much confusion because the terminology he used for describing his mathematical demonstrations is frequently different from the one we are currently using. Moreover, Frisch never provided a complete mathematical formulation for his PPIP model, neither in 1933 nor afterwards. In fact, Frisch only provided the main characteristics of his model; he also gave one example of application of his model which boils down to a single diagram entitled "cycles maintained by erratic shocks." Finally, Frisch provided a "mixed system of differential and difference equations" (now also known as delay differential equations). Such a model cannot be solved analytically in a general way. Therefore, several methods have been developed to approach the solution of such model (Cooke, 1963). Thank to such methods, four economists have provided numerical simulations of Frisch's model. However, depending on the method used, they have reached different conclusions. Thalberg (1990, p. 114) claimed that Frisch's PPIP model cannot fluctuate clearly with the original parameters and conditions. Following Thalberg's analysis, Zambelli (1992) studied Frisch's article and claimed that the "rocking horse model is not rocking" (1992, p. 52). Ricardo Duque (2009) argued that Zambelli's criticism can be solved when we consider Frisch's impulse mechanism in the simulation. Recently, Vincent Carret (2020) provided a computational analysis of Frisch's PPIP model by applying the Laplace transformation and its inverse. Therefore, contrary to Zambelli, Carret could claim that Frisch's model rocks.

Such contradictory conclusions and analyses show that some of the fundamental features of Frisch's analysis have not been completely highlighted. One of the main reasons for these different conclusions is that Frisch did not make explicit all the steps he took, leading the authors to interpret what he actually did. The fact that Frisch's 1933 publication was not an article published in a scientific journal contributes to explain such issues[1]. In addition, while Frisch

[1] Frisch's essay was published in the book *Economic Essays in Honor of Gustav Cassel* for which contributors were invited to write a chapter. In other words, Frisch did not have to follow the selection process as we have in a scientific journal, and he did not have to format his article according to the standards of a scientific article. Moreover, Frisch sent his

started his scientific career and scientific work as a mathematician (Bjerkholt, 2018, p. 13), Frisch's essay has been never investigated in a mathematical perspective[2]. These elements have led to misinterpret the contribution and the originality of this model, both at the mathematical. and the economic level. For instance, and as we detail in this paper, Thalberg, Zambelli, Duque and Carret have a common approach: to pass over the incompleteness of Frisch's model, they modified Frisch's model by adding hypotheses, changing the value of the parameters or using a different methodology. They argued that Frisch's models need to be extended in order to oscillate and to fit with their numerical results. In so doing, their extension of Frisch's models deviated from the original ones. Frisch's essay was so innovative that until now its whole contribution has not fully been appreciated.

2 Frisch's innovative approach revealed

Evaluating the originality of Frisch's PPIP model is a real challenge for two main reasons. First, as with any precursor, the terminology he used was not fixed yet at his time, and consequently nowadays we need to reinterpret some of Frisch's terms in order to read his essay correctly. Second, with his PPIP model, Frisch opened a new avenue in the theory of oscillations and so doing faced specific constraints which influenced his presentation in many ways. This section sheds light on the originally of Frisch's analysis and clarify what Frisch achieved.

Frisch (1933) used oscillation theory for analyzing business cycle, but at his time this theory was still in its infancy and its formalism was not completely fixed (Ginoux, 2017). Therefore, the definitions and terminologies used by Frisch are confusing and have generated many misinterpretations in the literature. The first confusing element concerns the classification of oscillations used by Frisch. He started his book chapter by claiming that "THE majority of the economic oscillations (...) seem to be explained most plausibly as free oscillations. In many cases they seem to be produced by the fact that certain exterior impulses hit the economic mechanism and thereby initiate more or less regular oscillations" (1933, p. 171). The meaning of the term "free oscillations" in Frisch's essay differs from the meaning of the term today. "Free oscillations" are now only associated with "undamped oscillations" or "harmonic oscillations", which have regular oscillations or more precisely *periodic* oscillations. In the second paragraph, Frisch clarified his conception, and we understand that "free oscillations" must be read as "free damped oscillations" in the whole essay. Another issue is the terminology used for describing the duration of the *business cycles*. Frisch (1933, pp. 171, 174, 186, 189) used sev-

chapter with a delay, and consequently it could not be reviewed by the editors as originally planned (Bjerkholt, 2007, pp. 473-474).

[2] It is worth reminding that Frisch met Divisia and other French mathematicians during his stay in Paris in 1921-1923. He also defended his doctorate on a subject in mathematical statistics rather than in economics (Bjerkholt, 2018, p. 13).

eral terms: the "length of the cycle", the "length of the period" and also the "period of the cycle". However, and as we will detail in the next section, the solutions of his model are "damped sine curves". Therefore, Frisch used the term "period of cycles" instead of "damped period of cycles". Consequently, we must replace Frisch's terms "length of the cycle", "length of the period" and "period of the cycle" with the expression "damped period" or "conditional period".

The use of the oscillation theory as a theoretical framework caused to Frisch a challenging issue: the statistical data he had access to show that business cycles did not have a regular period of recurrence (Akerman, 1928, p. 229; Hamburger, 1931, p. 27; Juglar, 1889; Mitchell, 1913, p. 581)[3]. In fact, economic oscillations appear to be maintained by the "rhythmical contraction and expansion of the economic system" (Hotelling, 1927, p. 289). How could Frisch model such observations with the oscillation theory of his time? In the beginning of the 1930s, there were only two paths for maintaining oscillations in a system: the first was *forced oscillations* investigated by Lord Rayleigh (1883) at the end of the nineteenth century; the second was *relaxation oscillations* introduced by Van der Pol (1925, 1926). Could Frisch consider relaxation oscillations? Relaxation oscillators, which belong to the class of self-maintained oscillators, are systems with a "nonlinear negative damping". For example, in the case of a damped pendulum this means that the damping is not a positive constant but depends nonlinearly on the amplitude of the oscillations. Thus, the damping can take either positive or negative values. In the latter, the motion is maintained instead of being damped. Moreover, one of the most important features of relaxation oscillators is that in such systems the amplitude and period are completely independent of the initial conditions (Ginoux, 2017, 109). Frisch did not follow this path which triggered out a "hunt for the relaxation effect" in the 1930s (Ginoux, 2017, p. 201). For modeling economic and cyclical oscillations, Frisch intended to design a system for which "the solution will depend, not only on the initial conditions of the system in a given *point* of time, as initial conditions we shall have to consider the shape of the curve over a whole interval of length ε" (1933, p. 183). In other words, Frisch imposed to his system to be dependent on the initial conditions and to model irregular maintained economic oscillations, which is neither compatible with one of the main characteristics of relaxation oscillations, nor with that of free or undamped oscillations. Therefore, Frisch could not consider relaxation oscillations for modeling business cycle.

In this situation, Frisch had to consider *forced oscillations*, which occur when an oscillating system is driven by a *periodic force* external to the system. For such class of oscillations, we can consider either forced undamped oscillations or *forced damped oscillations.* In the former case, if the *impressed frequency* of the *periodic force* is different from the *natural* (*undamped*) *frequency* of the system, oscillations will have the appearance of a regular pat-

[3] Frisch had also his own time series, but he did not provide them (Bjerkholt, 2007, p. 457).

tern[4]. Here again, this does not fit with the irregular economic fluctuations observed. Therefore, Frisch could not consider *forced undamped oscillations*. The last possibility was to consider forced damped oscillations. Here again Frisch faced to an issue. Indeed, at his time, the *forcing* was periodic and its amplitude constant. Both characteristics are incompatible with the irregularities observed in the business cycles.

To sum up, none of the options available at Frisch's time could be used for modeling the business cycles he observed. In fact, as we will highlight, Frisch's conception of economic and cyclical oscillations was deeply rooted in the paradigm of "free damped oscillations". So, to keep this paradigm and to model irregular economic maintained oscillations, Frisch proposed a model of *forced damped oscillations* with "irregular forcing" that he called "erratic shocks" or "impulses" which compensate the damping and maintain the oscillations. Such class of oscillations did not exist in 1933, so Frisch had to open a new avenue in the theory of oscillations. Specifically, Frisch's PPIP led to one of the most fruitful debates with another key econometrician, Michal Kalecki (1935)[5]. This debate started during the third *Econometric society* meeting that held in Leyden in 1933 and was published in *Econometrica* from 1935 onwards. This debate influenced some of the most important mathematicians and physicists who were developing the theory of oscillations. For instance, Andronov and Khaikin (1937) dedicated their chapter the "Theory of the clock" to this new type of oscillation; they renamed Frisch's new oscillations *impulse-excited oscillations*. During the Second World War, Minorsky (1942) investigated Self-excited Oscillations in Dynamical Systems Possessing Retarded Actions. Then, Minorsky (1947) who published the English translation of Andronov and Khaikin's book, popularized this terminology and defined this new kind of oscillation page 384. In the same vein, the French physicist Yves Rocard (1941), who also contributed to developing the theory of oscillations used the early work in macrodynamics for exploring new models in mathematics[6].

[4] In the particular case where the impressed frequency of the periodic force is equal to the natural (*undamped*) frequency of the system, amplitude of oscillations increase beyond all bounds as time passes. Such phenomenon is called *resonance*. As an example, in 1831, when an army was marching across a drawbridge in Britain, the drawbridge vibrated and the amplitude of the vibration grew rapidly. All soldiers were then thrown.

[5] We can also include Tinbergen who explained that "The first publication of this sort was a paper in Polish by Kalecki in *Proba Teorij Konjunktury*, Warszawa, 1933. A few months later appeared Frisch's "Propagation Problems and Impulse Problems". Both theories were presented at the Leyden meeting of the Econometric Society in 1933" (Tinbergen, 1935, p. 268)

[6] The influence of the early work in econometrics on the theory of oscillations developed in mathematics and physics would deserve a separate investigation. However, it is worth mentioning that Rocard proposed in 1941 an economic model based on the analyses of Frisch and Kalecki and on relaxation oscillations. Rocard's 1941 chaotic econometric model preceded Lorenz' butterfly of twenty-two years. Ginoux *et al.* (2022) established that this "new old" three-dimensional autonomous dynamical system is a new jerk system whose solution exhibits a chaotic attractor the topology of which varies, from a double scroll attractor to a Möbius-strip and then to a toroidal attractor, according to the values of a control parameter.

In order to clarify Frisch's new conception of *forced damped oscillations maintained by erratic shocks*, we can use the metaphor of the swing for children. In the first case, let's suppose that the child is alone. To swing, he has to stand on tiptoe far from the vertical stable position. Then, he starts swinging. Nevertheless, since the swing is damped because of air resistance and friction, after a while (*cycle*) the swing stops. This is exactly the case studied by Frisch (1933, p. 175) in the third and fourth sections of his article. In the second case, the child is still alone but he wants the swinging to be maintained. To this aim he moves his feet forward and backward when the swing reaches its maximal amplitude positions. This corresponds to *maintained* or *self-sustained oscillations* including the famous "relaxation oscillations'" of Van der Pol. In the third and last case, the child is with his father who pushes him in order to keep him swinging at constant amplitude (*impulses*). His father supplies energy to compensate the loss. More energy will be fed to the swing if his father pushes in the same direction as the velocity. This corresponds to the new avenue opened by Frisch (1933, p. 197) in the fifth section of his article entitled "Erratic shocks as a source of energy in maintaining oscillations".

For investigating this new avenue in the theory of oscillations, Frisch studied successively the different aspects of the propagation and impulse problems in business cycles. This point deserves a major clarification. To date, the literature has considered that Frisch's analysis in his book chapter like a solo model that is progressively complexified. Such interpretation is consistent with some of Frisch's sentences; however, it is not fully consistent with the mathematical hypotheses and formulations proposed by Frisch. Specifically, Frisch's purpose was to "approach the study of business cycle with the intention of carrying through an analysis that is truly dynamic ..." (Frisch, 1933, p. 172). It sounds trivial today, but from a historical standpoint, this is significant. Indeed, Frisch aimed at investigating an old problem in economics (namely business cycles) with a dynamic approach, which was totally new. To achieve this, he organized his essay in four main steps leading to four models. First step, in the third section of his book chapter, he exposed a "completely closed system", which is a mathematical *model without oscillations* (let's call it MODEL 1) from which he can determinate the trend of the business cycle. The second step consisted of providing a mathematical *model of damped oscillations with delay* (let's call it MODEL 2). This model is developed in the fourth section of Frisch's essay for solving the propagation problems. With this model, Frisch demonstrated that it was mathematically possible to find again the *damped period* of the observed economic business cycles. Having solved the propagation problem, Frisch discussed the impulse problem which was investigating in the last two sections (i.e., steps three and four). In section 5, called "Erratic shocks as a source of energy in maintaining oscillations", Frisch solved the impulse problems by introducing exogenous impulses that can maintain oscillations in a *model of free damped oscillations* (let's call this propagation and impulse model, MODEL 3). For doing this, Frisch suggested introducing exogenous shocks "impulses" and considered them as *random*. The introduction of these exogenous random shocks should have given birth to the famous "propagation-impulse model".

However, Frisch never provided an explicit mathematical formulation for his exogen "propagation-impulse model", neither in 1933 nor afterwards. In fact, and as Frisch mentioned (1933, p. 199), he only provided the main characteristics of his MODEL 3[7]. In the sixth and last section, called "The innovations as a factor in maintaining oscillations", Frisch discussed the economic framework for another "propagation-impulse model" based on Schumpeter's theory of innovations (let's call this propagation and impulse model, MODEL 4). To support his argument, Frisch provided a mechanical analogy of a pendulum and referred to "auto-maintained oscillations". In this perspective, the innovations are endogenous impulses which could maintain the oscillations. It is worth noting that Frisch never provided any mathematical formulation of his MODEL 4.

In the following sections, we will guide the readers in order to rebuild the complete mathematical PPIP's model that Frisch may have used to plot in his PPIP paper the figure 6 corresponding to oscillations maintained by erratic shocks.

3 Mathematical analysis of Frisch's propagation model

This section presents MODEL 1 and MODEL 2 that constitute what we can call Frisch's "propagation model". We will use these clarifications in the next section for rebuilding Frisch's MODEL 3.

3.1 MODEL 1: a simplified system without oscillations

At the economic level, MODEL 1 allowed Frisch to expose the relations between the macroeconomic variables in a stationary state (i.e., long-term equilibrium). At the mathematical level, this closed system enabled him to present the concept of characteristic exponents that were introduced by Henri Poincaré (1992 [1892, 1893, 1899], p. 176). The works of Poincaré were a common knowledge of all mathematicians at the time of Frisch. While Frisch did not refer to Poincaré, this section establishes for the first time that Poincaré's work allowed him to solve the ordinary differential equation of his MODEL 1.

By considering the annual investment, y, (called "the yearly production of capital goods") and the annual consumption, x, (called "the yearly production of consumer goods"), Frisch stated the following equation (the number in

[7] Frisch clarified that "For a more detailed mathematical analysis the reader is referred to a paper to appear in one of the early numbers of *Econometrica*" (Frisch, 1933, p. 199). Unfortunately, such a paper has never been published. While Frisch did not provide an explicit mathematical formulation for his "propagation-impulse model", his model is complete in the sense that Frisch found the values of the periods of the economic cycles by calibrating this model. Therefore, when Duque (2009, p. 48) argued that this model is incomplete, it must be understood that it is mathematically incomplete.

square brackets refers to the equation's number and the page in Frisch's book chapter)[8]:

$$y = mx + \mu\dot{x} \qquad (1)[\text{eq. 3, p. 177}]$$

where constant m expresses the depreciation of capital stock (called "the wear and tear on capital goods caused directly and indirectly by the production of one unit of consumption"), and μ "expresses the size of capital stock that is needed directly and indirectly in order to produce one unit of consumption per year" (Frisch, 1933, p. 177). Then, Frisch expressed the consumption's variation according to the annual consumption and the annual capital production:

$$\dot{x} = c - \lambda\left(rx + sy\right) \qquad (2)[\text{eq. 6, p. 180}]$$

"where c and λ are positive constants. The constant c expresses a tendency to maintain and perhaps expand consumption, while λ expresses the reining-in effect of the *encaisse désirée*" (1933, p. 180). The *encaisse désirée* $(rx + sy)$, which is the demand of cash needed for the transactions, is proportional to the annual investment and consumption. As a first approximation, Frisch considered "r and s as constants given by habits and by the nature of existing monetary institutions" (1933, p. 179).

These two equations form MODEL 1. It allows Frisch to establish a "secular trend" in the business cycle (1933, p. 180). It is worth reminding that, for a time series, the *secular variation* (sometimes called *secular trend* or *secular drift*) is the long-term non-periodic variation. A secular variation over a time scale of a century may be part of a periodic variation over a time scale of several centuries. Natural quantities often have both periodic and *secular variations*. Thus, Frisch considered that economic cycles may also have periodic and *secular variations*.

Frisch did not provide any demonstration for solving his MODEL 1. In the following paragraph, we provide a demonstration showing how Frisch used Poincaré's work. In so doing, we establish for the first time that Frisch based his analysis in his essay on the new methods of the celestial mechanics developed by Poincaré. Plugging Eq. (1) into Eq. (2) leads to the following equation:

$$\left(1 + \lambda\mu s\right)\dot{x} + \lambda\left(r + ms\right)x = c \qquad (3)$$

In order to solve this classical non-homogeneous first-order ordinary differential equation, Frisch used Poincaré's *characteristic exponents* which enable to state the stability of periodic solutions of a set of ordinary differential equations. Following Poincaré's method for computing the value of the characteristic exponents, Frisch proposes to search a solution of Eq. (3) in the form of

[8] The annual investment is determined by the accelerator principle (Dupont-Kieffer, 2012).

$$x\left(t\right)=a_{*}+ae^{\rho t} \tag{4}$$

where $\rho=-\beta+i\alpha$ and $i=\sqrt{-1}$. By differentiating Eq. (4) and replacing into Eq. (3), we obtain:

$$\left(1+\lambda\mu s\right)a\rho e^{\rho t}+a_{*}\lambda\left(r+ms\right)+\lambda\left(r+ms\right)ae^{\rho t}=c \tag{5}$$

Thus, "the characteristic equation is consequently of degree one, and has therefore only one single real root" (1933, p. 180). In fact, Eq. (5) leads to the following system:

$$\begin{cases}\left(1+\lambda\mu s\right)\rho+\lambda\left(r+ms\right)=0\\ a_{*}\lambda\left(r+ms\right)=c\end{cases} \tag{6}$$

which enables to state the solution of Eq. (3):

$$x\left(t\right)=\frac{c}{\lambda\left(r+ms\right)}+Ke^{-\lambda\frac{r+ms}{1+\lambda\mu s}t} \tag{7}$$

where K is a constant depending on the initial condition, i.e., $x(0)=x_0$. The first component of the Eq. (7) represents the equilibrium level, the second component represents the secular trend. Since all constants (c, r, m, s, λ) are positive, "we shall have a *secular trend* but no oscillations" (1933, p. 180, our italics).

Our analysis has shown that Frisch used Poincaré's methodology for solving his MODEL 1. The following section will establish how Poincaré's work is crucial for understanding the way Frisch solved his MODEL 2.

3.2 MODEL 2: a macro-dynamic system giving rise to oscillations

MODEL 1 cannot fit with the observed statistical data, therefore Frisch proposed a second model, a model with *damped oscillations with delay.* At the economic level, MODEL 2 allows him to account for the economic origin of the economic oscillations (with the delays in the variables). Frisch (1933, pp. 180-181) investigated the main economic theories of his time and kept Aftalion's hypothesis of a time-dependency in the variables that creates oscillations due to the carry-on-activity. The concept of carry-on activity, still use nowadays, reflects the idea that investments take time to be converted into new capital goods, which creates a lag "between capital goods ordered and capital goods delivered" (Frisch, 1931, p. 652). In this perspective, the capital production generates oscillations in the economy. By estimating the parameters of his model from statistical economic data, this enabled him, to compute the *characteristic coefficients* (the *damped period* (α) and the *damping exponent* (β)) of the *business cycles*, that is to say, of the "damped sine curves" solutions of his model. Thus, Frisch demonstrated that it was mathematically possible to find again the *damped period* of the observed economic business cycles.

3.2.1 Frisch's mixed system of differential and difference equations

In order to obtain a "dynamic system", Frisch added to the annual consumption, x, and the capital production, y, a third variable, z, which is "the carry-on-activity." He also took into account six parameters: m is the depreciation of capital, μ expresses the size of capital stock that is needed directly and indirectly in order to produce one unit of consumption per year, c expresses a tendency to maintain and perhaps expand consumption, λ expresses the reining-in effect of the *encaisse désirée*, r expresses consumption habits, and s is the nature of existing monetary institutions; and a constant ε.

Frisch (1933, p. 182) defined the carry-on activity as follows:

$$z\left(t\right) = \int_{\tau=0}^{\infty} D_\tau y\left(t-\tau\right) d\tau \qquad (8)[\text{eq. 1, p. 182}]$$

D_τ is "the amount of production activity needed at the period of time $t+\tau$ in order to carry on the production of a unit of capital goods started at the period of time t" (1933, pp. 181-182). In other terms, carry-on activity is time gap for producing capital goods, and it is a function of past investment decisions. Then, he explained:

"If D_τ is given as a simple mathematical expression, the system may under certain conditions be solved in explicit form. As an example, we shall assume that

$$D_\tau = \begin{cases} \dfrac{1}{\varepsilon} & 0<\tau<\varepsilon \\ 0 & \tau>\varepsilon \end{cases} \qquad (9)[\text{eq. 3, p. 182}]$$

where ε is a technically given constant" (1933, p. 182). ε is the number of units of time needed between the investment and its maturity.

Then, Frisch differentiated Eq. (8) with respect to time and obtained the variation of investment between t and $t+\varepsilon$:

$$\varepsilon\dot{z}\left(t\right) = y\left(t\right) - y\left(t-\tau\right) \qquad (10)[\text{eq. 4, p. 182}]$$

The proof of this computation, missing in Frisch's article, involves the so-called Leibniz-Newton theorem (Rudin, 1966):

$$\frac{dz\,(t)}{dt} = \frac{d}{dt}\int_{\tau=0}^{\infty} D_\tau y\,(t-\tau)\,d\tau = \int_{\tau=0}^{\infty} \frac{d}{dt}\left(D_\tau y\,(t-\tau)\right)d\tau$$
$$= \int_{\tau=0}^{\infty} D_\tau \frac{dy\,(t-\tau)}{dt}d\tau$$

Let's notice that: $\frac{dy\,(t-\tau)}{dt} = -\frac{dy\,(t-\tau)}{d\tau}$. So, we obtain:

$$\frac{dz\,(t)}{dt} = -\int_{\tau=0}^{\infty} D_\tau \frac{dy\,(t-\tau)}{d\tau}d\tau = -\frac{1}{\varepsilon}\int_{\tau=0}^{\tau=\varepsilon} \frac{dy\,(t-\tau)}{d\tau}d\tau$$
$$= -\frac{1}{\varepsilon}\left[y\,(t-\tau)\right]_{\tau=0}^{\tau=\varepsilon} = \frac{1}{\varepsilon}\left[y\,(t) - y\,(t-\varepsilon)\right]$$

Thus, Frisch could add Eq. (10) into his MODEL 1 and replaced y by z in the first equation of the system (i.e., the *encaisse désirée*) for obtaining his MODEL 2, which represents his "propagation model":

$$\begin{cases} \dot{x} = c - \lambda\,(rx + sz) \\ y = mx + \mu\dot{x} \\ \varepsilon\dot{z} = y\,(t) - y\,(t-\varepsilon) \end{cases} \qquad (11)\text{[eqs 2, 3.3, 4, p. 177 \& 182]}$$

Frisch's system (11) is a "mixed system of differential and difference equations" which cannot be solved analytically. However, according to Boumans (2005a, p. 57): "the general solution of a mixed difference-differential equation is an infinite weighted sum of harmonic functions". In other words, the solution can be given by an infinite sum of exponentials, which is defined by an initial movement on a certain period. As we will see below, Frisch first computed the *frequency* and *damping exponent* (see 3.2.3) and then, the *phase* and *amplitude* (see 3.2.4) of the four first *cyclical components* of the solution of his model (11). To this aim, Frisch needed first to estimate the value of the parameters of his model (11), i.e. MODEL 2 (see 3.2.2).

3.2.2 Frisch's estimation of parameter values of his model

Once he had established his MODEL 2 (11), Frisch focused on the parameter values involved in it. Frisch (1933, p. 185) explained:

> "In order to study the nature of the solutions, I shall now insert for the structural coefficients ε, μ, m, etc., numerical values that may in a rough way express the magnitudes which we would expect to find in actual economic life. At present I am only guessing very roughly at these parameters, but I believe that it will be possible by appropriate statistical methods to obtain more exact information about them (...). Let us first consider the constant ε. It expresses the total length of time needed for the completion of big units of fixed capital: big industrial

plants, water-power plants, railways, big steamers, etc. This span of time includes not only the actual time needed for the technical construction (the erection of the buildings, etc.) but also time needed for the planning and organization of the work (...). If three years is taken as the *average* lag of the various elements of production activity after the beginning of the planning, we shall have to put $\varepsilon = 6$ in (3), and consequently in (4), indeed in the case of equal distribution, as assumed in (3), the *average* lag will be half the *maximum* lag. Furthermore, let us put $\mu = 10$, which means that the total capital stock is ten times as large as the annual production. Further, let us put $m = 0.5$, which means that the direct and indirect yearly depreciation on the capital stock caused by its use in the production of the national income is one-half of that income, i.e. 20% of the capital stock. Finally, let us put $\lambda = 0.05$, $r = 2$, and $s = 1$. These latter constants, which represent the effect of the *encaisse désirée* on the acceleration of consumption, are of course inserted here by a still rougher estimate than the first constants. There is, however, reason to believe that these latter constants will not affect very strongly the length of the cycles obtained (see the computations below)."

As mentioned in his archives, Frisch did his own estimations, which were not published. However, it seems that it was extremely challenging for Frisch to calibrate his model (1933, p. 185). Thus, he used the following parameters set presented in the Table 1 below.

c	λ	r	s	m	μ	ε
0 - 0.165	0.05	2	1	0.5	10	6

Table 1 Parameters set of Frisch's system (11), from Frisch (1933, pp. 185-186).

Concerning the value of parameter c, Frisch will take either 0 or 0.165. In the following, we will use only $c = 0$.

3.2.3 Frequency and damping exponent of Frisch's model solution

As recalled by Cooke (1963, p. 165), *frequency* and *damping exponent* of a "mixed system of differential and difference equations" can be computed by using the so-called Poincaré's *characteristic exponents* that he introduced in his famous *New Methods of Celestial Mechanics* [36, Tome I, p. 162]. Thus, Frisch (1933, p. 183) investigates whether the system (11), i.e., his MODEL 2 is satisfied for "the variables x, y and z considered as times series of the form":

$$\begin{cases} x = a_* + \sum\limits_k a_k e^{\rho_k t} \\ y = b_* + \sum\limits_k b_k e^{\rho_k t} \\ z = c_* + \sum\limits_k c_k e^{\rho_k t} \end{cases} \qquad (12)[\text{eq. 8, p. 183}]$$

where ρ_k are complex or real constants, which are the so-called Poincaré's *characteristics exponents*[9], and where a, b and c are constants. Frisch (1933, p. 184) then showed "that all the must be the roots of the following characteristic equation":

$$\frac{\varepsilon\rho}{1 - e^{\varepsilon\rho}} = -\lambda s \frac{m + \mu\rho}{r\lambda + \rho} \qquad (13)[\text{eq. 10, p. 184}]$$

Then, Frisch considered that this equation (13) may have complex or real roots and he poses as previously (see Sect. 3.1), $\rho = -\beta + i\alpha$ and $i = \sqrt{-1}$ (Eq. 11, p. 184). This allowed him to separate the real and imaginary parts of the characteristic equation (13) and led him to the following "two equations to determine α and β (assuming ε and $\lambda\mu s \neq 0$)[10]":

$$1 + \lambda s \mu e^{\varepsilon\beta} \frac{\sin \varepsilon\alpha}{\varepsilon\alpha} = \frac{m \frac{\varepsilon^2}{\mu^2} (m - \lambda r \mu)}{\left(\varepsilon\beta - m \frac{\varepsilon}{\mu}\right)^2 + (\varepsilon\alpha)^2} \qquad (14a)[\text{eq. 12, p. 184}]$$

$$-\frac{\varepsilon\beta - \lambda r \varepsilon + m \frac{\varepsilon}{\mu}}{\varepsilon\beta - m \frac{\varepsilon}{\mu}} + \lambda s \mu \frac{1 - e^{\varepsilon\beta} \cos \varepsilon\alpha}{\varepsilon\beta - m \frac{\varepsilon}{\mu}} = \frac{m \frac{\varepsilon^2}{\mu^2} (m - \lambda r \mu)}{\left(\varepsilon\beta - m \frac{\varepsilon}{\mu}\right)^2 + (\varepsilon\alpha)^2} \qquad (14b)[\text{eq. 13, p. 184}]$$

The system (14) consisting of two transcendental equations (14a-14b) cannot be solved analytically except by making use of approximations as Frisch and his assistants had done (1933, p. 184)[11]. However, the roots of the *characteristic equation* (13) as well as those of equations (14a-14b) can be obtained graphically, like Frisch did with his assistant although he did not reproduce it in his essay (Duque, 2009). Nowadays modern software such as *Mathematica* can computed numerically them, as we will do in order to improve Frisch's results. First, Frisch (1933, p. 184) explained:

> "In other words, if complex roots occur they will be conjugate, which means that the corresponding component of x, y and z will actually be a damped sine curve."

He investigated "if there exists a magnitude β such that $(\beta, 0)$ is a root". So, Eq. (13) had been plotted in Fig. 1 for $\alpha = 0$ and so for $\rho = -\beta$.

[9] See Sect. 3.1, Eq. (4).

[10] We can notice that α is necessary non-null otherwise the system would not oscillate.

[11] For more details about Frisch's method of resolution of system (14), see Duque (2009).

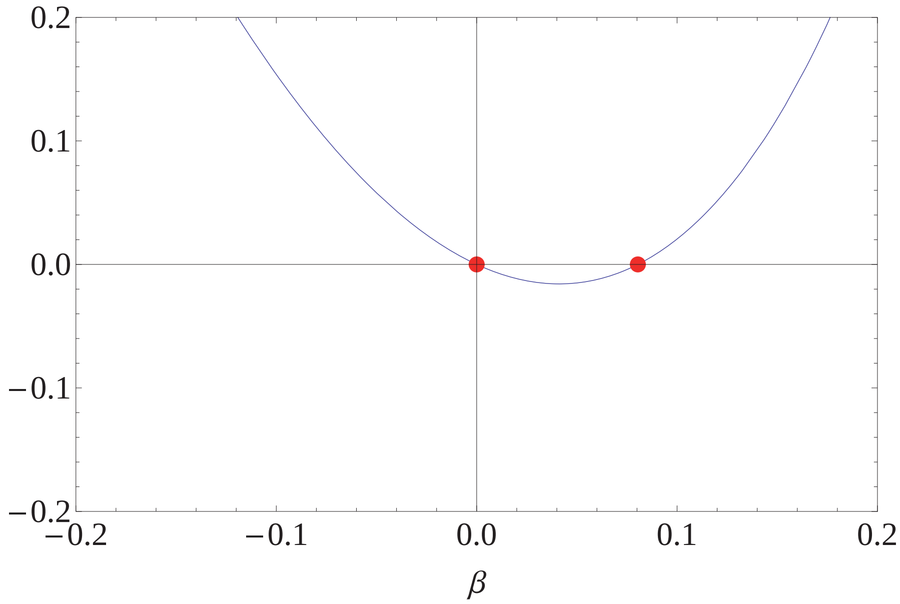

Fig. 1 Graphical representation of the root of Eq. (13).

Let us notice that the roots of Eq. (13) can be graphically estimated from the Fig. 1. We observe a first root for $(\beta, \alpha) = (0, 0)$ and a second one for $(\beta, \alpha) \approx (0.1, 0)$ (red points in Fig. 1). By using the function *FindRoot* of *Mathematica* this root can be numerically computed with high accuracy. We found that $\beta_0 = 0.0804385$ (Frisch found $\beta_0 = 0.08045$) and since $\rho = -\beta$, we have:

$$\rho_0 = -0.0804385$$

Then, the equations (14a-14b) have been plotted in Fig. 2. Eq. (14a) has been plotted in blue and Eq. (14b) in red. The points of intersection of these two curves in the shape of blue and red tongues provide the roots (α, β) of Eqs. (14a-14b). As previously, these roots can be graphically estimated from the Fig. 2. As an example, the first root is $(\alpha, \beta) \approx (0.7, 0.4)$.

Still using the function *FindRoot* of *Mathematica* all these roots can be numerically computed with high accuracy. The results are presented in Tab. 2.

These roots are the exact same as those computed by Frisch and his assistants as can be seen in Frisch's Table 2 reproduced below (see Fig. 3). They correspond to what Frisch (1933, p. 187) called the *trend*, the *primary cycle*, the *secondary cycle* and the *tertiary cycle*[12].

[12] As Duque (2009, p. 23) explained "these values were the result of endless calculations and trial-and-error attempts as evidenced by his notes taken while preparing the manuscript".

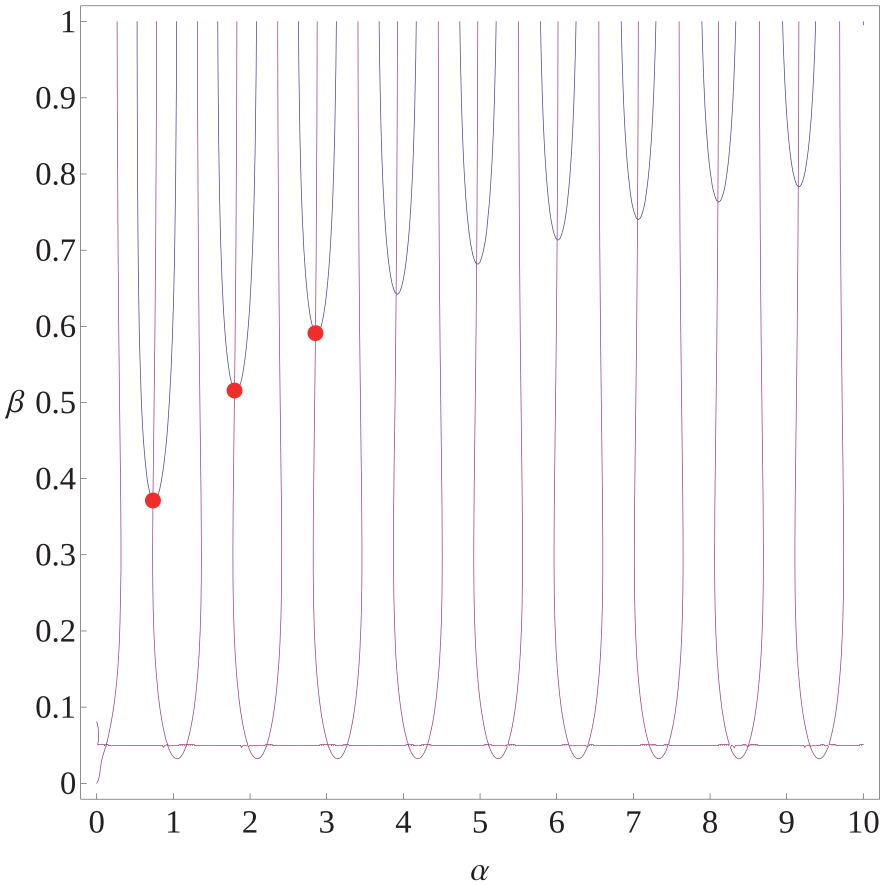

Fig. 2 Graphical representation of the roots of Eqs. (14a-14b).

TABLE I

CHARACTERISTIC COEFFICIENTS OF THE COMPONENTS OBTAINED

	Trend ($j=0$)	Primary Cycle ($j=1$)	Secondary Cycle ($j=2$)	Tertiary Cycle ($j=3$)
Frequency .. .. α	$\rho_0 = -0{\cdot}08045$	0·73355	1·79775	2·8533
Period .. .. $p = \frac{2\pi}{\alpha}$		8·5654	3·4950	2·2021
Damping exponent .. .. .. β		0·371335	0·5157	0·59105
Damping factor per period .. .. $e^{-2\pi\beta/\alpha}$		0·0416	0·1649	0·2721

Fig. 3 Characteristic coefficients of the components obtained from Frisch (1933, p. 187).

	Root 0 ($j=0$)	Root 1 ($j=1$)	Root 2 ($j=2$)	Root 3 ($j=3$)
α_j	0	0.733552	1.79772	2.8533
β_j	0.0804385	0.371354	0.515701	0.591061

Table 2 Numerical values of the roots of Eq. (13) and Eqs. (14a-14b).

According to Frisch (1933, p. 188):

"Besides the secular trend, there will be a primary cycle with a period of 8.57 years, a secondary cycle with a period of 3.50 years, and a tertiary cycle with a period of 2.20 years [see Fig. 3]. These cycles are determined by the first, second and third set of conjugate complex roots of [13]. These sets are denoted $j = 1, 2, 3$ in [Fig. 3]. There will also be shorter cycles corresponding to further roots of [13], but I shall not discuss them here."

Of course, as highlighted in Fig. 2, Eqs. (14a-14b) have more than three roots. Frisch only focused on the first three roots of Eqs. (14a-14b) and found three *cycles* for $j = 1$, 2 and 3. Frisch (1933, pp. 188-189) pointed out that the calibration of his model allowed him to obtain "the length of the primary cycle of 8½ years and the secondary cycle of 3½ years", which is "not entirely due to coincidence" (1933, p. 191).

3.2.4 Phase and amplitude of Frisch's model solution

According to Boumans (2005a, p. 57), *phase* and *amplitude* ("weight") of the solution to Frisch's system (11) is given by an infinite sum of exponentials (12), which is defined by an initial movement on a certain period. By replacing k by j and by posing $\rho_j = -\beta_j + i\alpha_j$, Frisch's equations (12), i.e., the solution of his system (11), i.e., of MODEL 2, can be rewritten as follows:

$$\begin{cases} x(t) = a_* + \sum_j a_j e^{\rho_j t} = a_* + \sum_j a_j x_j(t) \\ y(t) = b_* + \sum_j b_j e^{\rho_j t} = b_* + \sum_j b_j y_j(t) \\ z(t) = c_* + \sum_j c_j e^{\rho_j t} = c_* + \sum_j c_j z_j(t) \end{cases}$$

where the "cyclical components" read:

$$\begin{cases} x_j(t) = A_j e^{-\beta_j t} \sin(\varphi_j + \alpha_j t) \\ y_j(t) = B_j e^{-\beta_j t} \sin(\psi_j + \alpha_j t) \\ z_j(t) = C_j e^{-\beta_j t} \sin(\theta_j + \alpha_j t) \end{cases} \qquad (15)\text{[eq. 18, p. 190]}$$

Thus, Frisch computed the *phase* and *amplitude* of the four first *cyclical components* of the solution (15) of his model (11), i.e. $j = 0$, 1, 2 and 3. In order to simplify and also to reduce the length of our demonstration, we will only focus on the *primary cycle* of (15), i.e. $j = 1$. Frisch (1933, p. 192) explained that the *primary cycle*, solution of his system (11), is represented by the functions:

$$\begin{cases} x_1(t) = 0.6816e^{-\beta_1 t}\sin(\alpha_1 t) \\ y_1(t) = 5.4585e^{-\beta_1 t}\sin(1.9837 + \alpha_1 t) \\ z_1(t) = -10.6662e^{-\beta_1 t}\sin(1.9251 + \alpha_1 t) \end{cases} \qquad (16)[\text{eq. 23b, p. 192}]$$

Starting from the first equation of (15) and by considering that $j = 1$ (*primary cycle*), we have:

$$x_1(t) = A_1 e^{-\beta_1 t}\sin(\varphi_1 + \alpha_1 t)$$

But according to Frisch (1933, pp. 183, tab.181): $\varphi_1 = 0$ and $A_1 = 1/2\alpha_1$. Moreover, Frisch (1933, pp. tab.1, 183) computed the values of α_1 and β_1 which have been provided in Fig. 3 and Tab. I (our computation). So, with $\alpha_1 = 0.733552$, we have $A_1 = 0.6816$ and then $x_1(t) = 0.6816e^{-\beta_1 t}\sin(\alpha_1 t)$. By using the second equation of the system (11), i.e., $y = mx + \mu\dot{x}$, we obtain that:

$$\begin{aligned} y_1 &= mx_1 + \mu\dot{x_1} \\ &= 0.5\left[0.6816e^{-\beta_1 t}\sin(\alpha_1 t)\right] + 10\left[-0.2531e^{-\beta_1 t}\sin(\alpha_1 t) + 0.4999e^{-\beta_1 t}\cos(\alpha_1 t)\right] \end{aligned}$$

since $\dot{x_1} = -0.2531e^{-\beta_1 t}\sin(\alpha_1 t) + 0.4999e^{-\beta_1 t}\cos(\alpha_1 t)$ with $\beta_1 = 0.371335$. This leads to:

$$y_1 = \left[-2.1902\sin(\alpha_1 t) + 5\cos(\alpha_1 t)\right]e^{-\beta_1 t}$$

Thus, with the classical method of normalization, we find that $\sqrt{(-2.1902)^2 + 5^2} = 5.4585$. By posing: $\cos\phi = -2.1902/5.4585$ and $\sin\phi = 5/5.4585$, we deduce that $\phi = 1.9837$ (in radians unit). By using the trigonometric formula: $\sin a\cos b + \sin b\cos a = \sin(a + b)$, we have:

$$y_1 = 5.4585e^{-\beta_1 t}\sin(1.9837 + \alpha_1 t)$$

So, $y_1(t)$ is indeed a solution of Frisch's system (11). This proof can be extended to all other variables and for all cycles. The plots of all variables (x_j, y_j, z_j) for $j = 1$, 2, 3 provide the same results as those obtained by Frisch (1933, pp. 192, eq. 23a, 23b, 23c, 23d) reproduced in Fig. 4 below.

$$(23a)\quad \begin{cases} x_0 = 1{\cdot}32 - 0{\cdot}32e^{-0{\cdot}08045t} \\ y_0 = 0{\cdot}66 + 0{\cdot}09744e^{-0{\cdot}08045t} \\ z_0 = 0{\cdot}66 + 0{\cdot}12512e^{-0\ 08045t} \end{cases}$$

$$(23b)\quad \begin{cases} x_1 = 0{\cdot}6816e^{-\beta_1 t}\sin \alpha_1 t \\ y_1 = 5{\cdot}4585e^{-\beta_1 t}\sin(1{\cdot}9837 + \alpha_1 t) \\ z_1 = -\ 10{\cdot}662e^{-\beta_1 t}\sin(1{\cdot}9251 + \alpha_1 t) \end{cases}$$

$$(23c)\quad \begin{cases} x_2 = 0{\cdot}27813e^{-\beta_2 t}\sin \alpha_2 t \\ y_2 = 5{\cdot}1648e^{-\beta_2 t}\sin(1{\cdot}8243 + \alpha_2 t) \\ z_2 = -\ 10{\cdot}264e^{-\beta_2 t}\sin(1{\cdot}7980 + \alpha_2 t) \end{cases}$$

$$(23d)\quad \begin{cases} x_3 = 0{\cdot}17524e^{-\beta_3 t}\sin \alpha_3 t \\ y_3 = 5{\cdot}0893e^{-\beta_3 t}\sin(1{\cdot}7582 + \alpha_3 t) \\ z_3 = -\ 10{\cdot}147e^{-\beta_3 t}\sin(1{\cdot}7412 + \alpha_3 t) \end{cases}$$

Fig. 4 Frisch's "cyclical components" $x_j(t), y_j(t), z_j(t)$ for $j = 0$, 1, 2 and 3, from Frisch (1933, p. 192).

3.2.5 Frisch's step-by-step computation for the primary cycle

In this subsection we reconstruct Frisch's step-by-step computation for the primary cycle. To this aim, we prove how he obtained the values of the solution $(x_1(t), y_1(t), z_1(t))$ for the primary cycle at $t = 0, 1/6, 2/6, 3/6, 4/6, \ldots$ presented in his Table 2 (Frisch 1933, p. 196).

Frisch (1933, p. 196) explained the method of successive numerical approximation in the following words:

> "In order to show this, we take for granted that the time shape of one of the curves – for instance, y_1 – is known in the interval $-6 < t < 0$. That is to say, in this interval we simply consider the values of y_1 as given by the expression [(16) [23b, p. 192]]. Then we want by the dynamic equations to determine the solutions numerically from the point $t = 0$ and onwards."

Then, Frisch inserted in his system (11), i.e., MODEL 2 the parameters set of Tab. 1 and obtained:

$$\begin{cases} y(t) = 0.5x(t) + 10\dot{x}(t) \\ \dot{x}(t) = -0.1x(t) - 0.05z(t) \\ -6\dot{z}(t) = y(t-6) + 0.5(x(t) + z(t)) \end{cases} \quad (17)\text{[eq. 24 p. 196]}$$

Frisch (1933, p. 196) specified:

"We shall use [eq. (17)] for a step-by-step computation. Since $x = 0$ and $\dot{x} = 0.5$ are given at the origin, $z = -10$ may be determined from the second equation in [eq. (17)]. Furthermore, since y_{t-6} is given, we may compute z in origin by means of the third equation in [eq. (17)], and finally y may be computed by the first equation in [eq. (17)]. Thus, we have all the items in the first line of Table 2. Since we know x and $\dot{x}$ in origin, we may by a straight linear extrapolation determine x and z in the next point of time, that is to say, in the second line of Table 2. And knowing x and z in this point, we may from the second equation of [eq. (17)] compute $\dot{x}$. Further, taking the value of y_{t-6} as given also in the next line we can compute $\dot{z}$, etc. In this way we may continue from line to line and determine the development of all the three variables x, y and z."

- For $t = 0$, $x(0) = 0$ and $\dot{x}(0) = 0.5$ according to Frisch. Moreover, he explained above that "we simply consider the values of y_1 as given by the expression [eq. (16)]". So, we have: $y(t) = y_1(t) = 5.4585e^{-\beta_1 t} \sin(1.9837 + \alpha_1 t)$. Thus, we can compute easily $y(-6) = 5.4585e^{6\beta_1} \sin(1.9837 - 6\alpha_1) = -33.5581$ which corresponds to the value given by Frisch (1933, p. 196) in the last column of his Table 2 (see Fig. 5 below).

t	x	y	z	$\dot{x}$	$\dot{z}$	y_{t-6}
0	0	5·0000	− 10·000	0·5000	6·4264	− 33·5581
0·16667	0·08333	4·4229	− 8·929	0·4381	6·6811	− 35·6634
0·33333	0·15635	3·8297	− 7·8155	0·3752	6·7865	− 36·8894
0·50000	0·21887	3·2329	− 6·6844	0·3124	6·7592	− 37·3221
0·66667	0·27093	2·6435	− 5·5579	0·2508	6·6153	− 37·0480

Fig. 5 Frisch's step-by-step computation of the primary cycle, from Frisch (1933, p. 196).

Let's prove how Frisch obtained these values. By using the second equation of (17) [eq 24, p. 196], we have:

$$\dot{x}(0) = -0.1x(0) - 0.05z(0) \tag{18}$$

Given that $x(0) = 0$ and $\dot{x}(0) = 0.5$, we obtain:

$$z(0) = -10 \tag{19}$$

The third equation of (17) [eq 24, p. 196] gives:

$$-6\dot{z}(0) = y(-6) + 0.5(x(0) + z(0)) \tag{20}$$

Since, $y(-6) = -33.5581$ is given, we find:

$$\dot{z}(0) \approx 6.4264 \tag{21}$$

Finally, the first equation of (17) [eq 24, p. 196] provides:

$$y(0) = 0.5x(0) + 10\dot{x}(0) \tag{22}$$

We deduce that:

$$y(0) = 5 \tag{23}$$

Following Frisch's indication "we may by a straight linear extrapolation determine x and z in the next point of time." In fact, it will be established in what follows that Frisch's straight linear extrapolation or Frisch's step by step computation is nothing else but the Euler method (Fröberg, 1985, p. 322) for which:

$$\dot{u}(t) = \frac{u(t+h) - u(t)}{h} \tag{24}$$

where h represents the step size. Frisch used a step size $h = 1/6$ as it may be easily deduced from the first column of his Table 2 (see Fig. 4 above).

- For $t = 1/6$, according to Frisch, we can deduce the value of $x(1/6)$ from that of $\dot{x}(0) = 0.5$. We have:

$$\dot{x}(0) = \frac{x\left(0 + \frac{1}{6}\right) - x(0)}{\frac{1}{6}} = 0.5 \tag{25}$$

We find thus:

$$x\left(\frac{1}{6}\right) = \frac{1}{12} = 0.08333 \tag{26}$$

Since we have found just above that $\dot{z} = 6.4264$, we have:

$$\dot{z}(0) = \frac{z\left(0 + \frac{1}{6}\right) - z(0)}{\frac{1}{6}} = 6.4266 \tag{27}$$

from which we deduce that

$$z\left(\frac{1}{6}\right) = -8.929 \tag{28}$$

Then, by using the second and the first equations of (17) [eq 24, p. 196], we have:

$$\dot{x}\left(\frac{1}{6}\right) = -0.1x\left(\frac{1}{6}\right) - 0.05z\left(\frac{1}{6}\right) = 0.4381 \tag{29}$$

and

$$y\left(\frac{1}{6}\right) = 0.5x\left(\frac{1}{6}\right) + 10\dot{x}\left(\frac{1}{6}\right) = 4.4226 \tag{30}$$

The values $x(1/6)$, $y(1/6)$, $z(1/6)$ and $\dot{x}(1/6)$ are exactly those obtained by Frisch (1933, p. 196) (see second line of his Table 2 reproduced in Fig. 5 above). As Frisch explained (1933, p. 197):

> "In this way we may continue from line to line and determine the development of all the three variables x, y and z."

In order to "give an idea of the closeness of the approximation obtained by the numerical step-by-step solution" Frisch (1933, p. 197) compared his results (presented just above and summarized in his Table 2 reproduced in Fig. 4) with those "determined by the explicit formula (23b)", that is to say with the equations (15) in which t takes the values $t = 0, \frac{1}{6}, \frac{1}{3}, \ldots$. Frisch (1933, p. 196) presented the results in his Table 3 and highlighted some slight differences with the values of his Table 2 as it may be observed.

3.2.6 Frisch's rocking horse model is rocking

As we have explained in the introduction, according to some authors, in particular Zambelli (1992), it has not been clearly established that Frisch's system (11), i.e., MODEL 2 does indeed rock. This debate has not been resolved to date. This subsection will demonstrate that Frisch's system (11), i.e., MODEL 2 does rock. As recalled just above, the solution of Frisch's model (11) is given by the sum of an infinite number of exponentials equal defined by equations (12) which can be rewritten as follows:

$$\begin{cases} x(t) = a_* + \sum_j a_j x_j(t) = a_* + a_0 x_0(t) + a_1 x_1(t) + a_2 x_2(t) + a_3 x_3(t) + \ldots \\ y(t) = b_* + \sum_j b_j y_j(t) = b_* + b_0 y_0(t) + b_1 y_1(t) + b_2 y_2(t) + b_3 y_3(t) + \ldots \\ z(t) = c_* + \sum_j c_j z_j(t) = c_* + c_0 z_0(t) + c_1 z_1(t) + c_2 z_2(t) + c_3 z_3(t) + \ldots \end{cases} \tag{31}$$

where the "cyclical components" $x_j(t), y_j(t), z_j(t)$ for $j = 0$, 1, 2 and 3 are given by equations (23a, 23b, 23c & 23d) in Frisch (1933, p. 192) and reproduced in Fig. 4 above.

In his demonstration, Zambelli (1992) considered the "aggregated evolution of each variable", i.e., the "sum of each individual component". As an example, he expressed $x(t)$ as follows:

$$x\left(t\right) = x_0\left(t\right) + x_1\left(t\right) + x_2\left(t\right) + x_3\left(t\right) + \dots \tag{32}$$

Then, he plotted $x(t)$ (also $y(t)$ and $z(t)$) and claimed that:

> "The proper procedure would be to decompose in harmonics the evolution of the aggregated magnitudes, x_j, y_j, z_j that took place in the past (...). In so doing it should be possible to derive an explicit solution for SYSTEM1 [MODEL 2 here]" (1992, p. 41).

Then, Zambelli (1992, p. 41) reached the conclusion that:

> "But even when this has been done one needs to investigate the qualitative behaviour of the system. The explicit solution may, of course, exhibit monotonic or non cyclical behaviour."

Nevertheless, there is a main drawback in Zambelli's procedure: he did not normalize the coefficients a_j, b_j and c_j. In other words, in the above expression (32) of $x(t)$, Zambelli posed $a_0 = a_1 = a_2 = a_3 = 1$. While in his publication Frisch (1933, p. 191) insisted on the fact that:

> "any linear combination of (15) [eq. 18 p. 190] (with constant coefficients) satisfy the dynamic system provided only that the sum of the coefficients by which they are linearly combined is equal to unity.

Unfortunately, it does not seem that Zambelli has taken into account such normalization[13] since in his case $a_0 + a_1 + a_2 + a_3 = 4$. If we follow Frisch's recommendations (namely that the sum of the coefficients must be equal to unity), the solution of Frish's system (11), i.e., of MODEL 2 does rock. By using expression (31), the solution $x(t)$ of Frisch's system (11), i.e. of MODEL 2 can be written as follows:

$$x\left(t\right) = a_0 x_0\left(t\right) + a_1 x_1\left(t\right) + a_2 x_2\left(t\right) + a_3 x_3\left(t\right) + \dots \tag{33}$$

This solution $x(t)$ has been plotted in Fig. 6 below with the following parameters: $a_0 = 0.01$, $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.699$, the sum of which is obviously equal to unity, as recommended by Frisch. We observe in Fig. 6 that the solution $x(t)$, i.e., the linear combination of the *trend*, *primary*, *secondary*, and *tertiary cycles*, exhibits *free damped oscillations* in Frisch's terminology or *damped oscillations* in the current one. Thus, it follows that Frisch's rocking horse model does rock, for a while of course, but it does rock.

[13] For an extended analysis of Zambelli's error, see Jovanovic and Ginoux (2020).

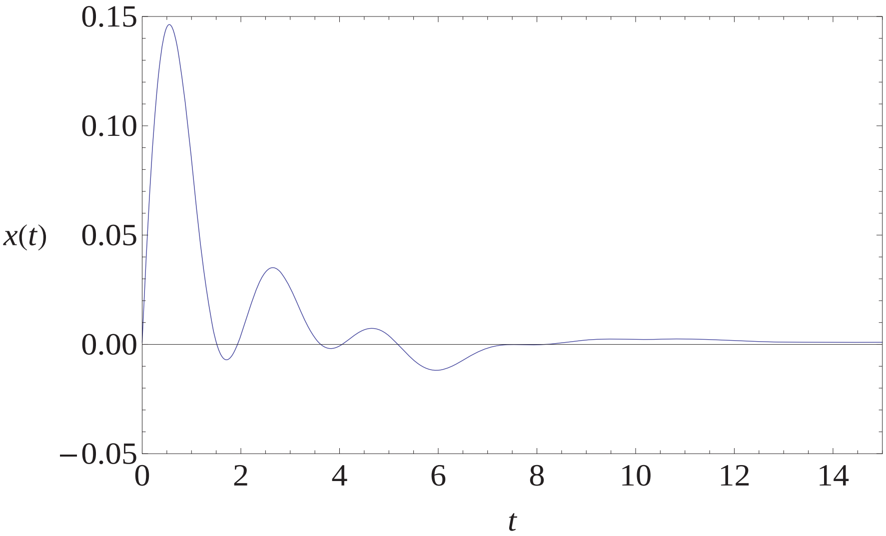

Fig. 6 Solution $x(t)$ of Frish's system (11), i.e., of MODEL 2.

4 A mathematical analysis of Frisch's propagation-impulse model

Frisch's MODEL 2 solves the propagation problem. Now let's turn to the second aspect of Frisch's analysis, the impulse problem associated with the propagation problem. In his section 5, Frisch aimed at solving the impulse problems by introducing exogenous shocks that can maintain oscillations in his *free damped oscillation* model. He used results from Eugen Slutsky (1927), Udny Yule (1927), Harold Hotelling (1927). These authors demonstrated that irregular fluctuations may be transformed into regular fluctuations similar to business cycles. In other words, behind the irregularity we observe, a regularity can be established. Such a regularity paves the way to a universal model of business cycles. Frisch called these exogenous shocks "impulses" and considered them as *random*. The introduction of these exogenous random shocks in a dissipative system should have given birth to the famous "propagation-impulse model" (MODEL 3). However, Frisch never provided a complete mathematical formulation of this model; he only gave its main characteristics[14]. In this section, we will analyze Frisch's draft of the PPIP model. We will highlight its two major drawbacks which precluded Frisch to provide a "universal" model as he intended to conceive it.

[14] Frisch (1933, 199) referred to a paper to appear in *Econometrica*, but such a paper has never been published.

4.1 Frisch's draft of his PPIP model

In MODEL 2, with damped oscillations, the business cycles disappear after a certain period of time. With his PPIP model, Frisch aimed at showing that it is possible to maintain oscillations by erratic shocks. In other words, such erratic shocks supply to the system the energy necessary to compensate the damping and maintain the observed oscillations. In order to present the impulse phenomena, Frisch based his explanation on "an oscillating pendulum whose movement is hampered by friction" (1933, p. 199):

$$\ddot{y} + 2\beta\dot{y} + \left(\alpha^2 + \beta^2\right) y = 0 \qquad (34)[\text{eq. 1, p. 199}]$$

where y represents the angular deviation of the pendulum from its vertical equilibrium position and "β and α are positive constants, expressing the strength of the friction" (1933, p. 199). By following the same approach as the one presented in section 3.2.4, it is easy to show that Frisch chose the coefficients of this second-order ordinary differential equation representing "damped oscillations" (34) such that α exactly corresponds to the "damped period". In fact, the discriminant of the *characteristic equation* reads:

$$\Delta = (2\beta)^2 - 4\left(\alpha^2 + \beta^2\right) = -4\alpha^2$$

Since, this discriminant is negative, the general solution of (34) is exactly the same as (15) – namely his MODEL 2 – as recalled by Frisch (1933, 199):

$$He^{-\beta t} \sin\left(\varphi + \alpha t\right) \qquad (35)$$

"where the amplitude H and the phase φ are determined by the initial conditions."

In order to "see immediately how the initial conditions determine the curve," Frisch (1933, 200) showed that the solution (35) may be written in the form

$$y\left(t\right) = P\left(t - t_0\right) y_0 + Q\left(t - t_0\right) \dot{y}_0 \qquad (36)[\text{eq. 2, p. 200}]$$

where $P(\tau)$ and $Q(\tau)$ are two functions independent of the initial conditions and defined by

$$P\left(\tau\right) = \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha} e^{-\beta\tau} \sin\left(\upsilon + \alpha\tau\right) \qquad (37)[\text{eq. 3, p. 200}]$$

$$Q\left(\tau\right) = \frac{1}{\alpha} e^{-\beta\tau} \sin\left(\alpha\tau\right) \qquad (38)[\text{eq. 4, p. 200}]$$

where

$$\sin\left(\upsilon\right) = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \qquad \cos\left(\upsilon\right) = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \qquad (39)[\text{eq. 5, p. 200}]$$

By using the trigonometric formula: $\sin a \cos b + \sin b \cos a = \sin(a+b)$, it is easy to state Eq. (36) [eq. 2, p. 200]. Starting from Eq. (34) [eq. 1, p. 199], the solution of this second order linear homogeneous ordinary differential equation is well known and can be written as:

$$y(t) = e^{-\beta t}\left[\lambda \cos(\alpha t) + \mu \sin(\alpha t)\right]$$

Its time derivative reads:

$$\dot{y}(t) = -\beta e^{-\beta t}\left[\lambda \cos(\alpha t) + \mu \sin(\alpha t)\right] + e^{-\beta t}\left[-\alpha\lambda \sin(\alpha t) + \alpha\mu \cos(\alpha t)\right]$$

Thus, we deduce that:

$$\begin{aligned} y(0) &= \lambda \\ \dot{y}(0) &= \mu\alpha - \beta\lambda \end{aligned}$$

This leads to:

$$\begin{aligned} \lambda &= y(0) \\ \mu &= \frac{1}{\alpha}\dot{y}(0) + \frac{\beta}{\alpha}\lambda = \frac{1}{\alpha}\dot{y}(0) + \frac{\beta}{\alpha}y(0) \end{aligned}$$

By replacing in the solution $y(t)$, we obtain:

$$y(t) = \frac{e^{-\beta t}}{\alpha}\left[\alpha \cos(\alpha t) + \beta \sin(\alpha t)\right] y(0) + \frac{e^{-\beta t}}{\alpha}\sin(\alpha t)\,\dot{y}(0)$$

Thus, with the classical method of normalization, we find that:

$$y(t) = \frac{\sqrt{\alpha^2+\beta^2}}{\alpha}e^{-\beta t}\left[\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cos(\alpha t) + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\sin(\alpha t)\right] y(0) + \frac{e^{-\beta t}}{\alpha}\sin(\alpha t)\,\dot{y}(0)$$

By using Eq. (39) [eq. 5, p. 200] and the formula: $\sin a \cos b + \sin b \cos a = \sin(a+b)$, we have:

$$y(t) = \frac{\sqrt{\alpha^2+\beta^2}}{\alpha}e^{-\beta t}\sin(\upsilon + \alpha t)\, y(0) + \frac{e^{-\beta t}}{\alpha}\sin(\alpha t)\,\dot{y}(0)$$

By using Eq. (37) [eq. 3, p. 200] and Eq. (38) [eq. 4, p. 200], we find again the expression (36) [eq. 2, p. 200] for the particular case where $t_0 = 0$. Then, a simple translation enables to obtain the result of Frisch.

Then, Frisch (1933, pp. 200-201) presented his concept of "impulses" i.e., the maintenance of damped oscillations by erratic shocks, as follows:

> Suppose that the pendulum starts with the specified initial conditions at the point of time to and that it is hit at the points of time t_1, t_2, $\ldots t_n$ by shocks which may be directed either in the positive or in the negative sense and that may have arbitrary strengths. Let y_k and $\dot{y}_k$

> be the ordinate and the velocity of the pendulum immediately before it is hit by the shock number k. The ordinate y_k is not changed by the shock, but the velocity is suddenly changed from y_k to $\dot{y}_k + e_k$, where e_k is the strength of the shock; mechanically expressed it is the quantity of motion divided by the mass of the pendulum. [W]e can consider $\dot{y}_k$ and e_k as two independent contributions to the later ordinates of the variable. In other words, the fact of the shock may simply be represented by letting the original pendulum move on undisturbed but letting a new pendulum start at the point of time t_k with an ordinate equal to zero and a velocity equal to e_k. This argument may be applied to all the points of time. We simply have to start in each of the points of time $t_1, t_2, \ldots t_n$ a new pendulum with an ordinate equal to zero and a velocity equal to the strength of the shock occurring at that moment, and then let all these pendulums continue their undisturbed motion into the future. The sum of the ordinates of all these pendulums at any given point of time t will then be the same as the ordinate $y(t)$ of a single pendulum which has been subject to all the shocks. In other words, the ordinate $y(t)$ will simply be
>
> $$y\left(t\right) = P\left(t - t_0\right) y_0 + Q\left(t - t_0\right) \dot{y}_0 + \sum_{k=1}^{n} Q\left(t - t_k\right) e_k \quad (40)[\text{eq. 7, p. 200}]$$

By analogy with the pendulum, this equation shows that the level of the Gross domestic product (GDP) depends of the initial condition y_0 and its kind of weighting coefficient, $P(t - t_0)$, the initial velocity $\dot{y}_0$ and its kind of weighting coefficient $Q(t - t_0)$, and the cumulated strength of all past shocks.

In order to obtain the formulation of his PPIP model, Frisch simplified Eq. (40). Considering that the more distant the past shocks are from the current situation, $y(t)$, the less important their impact on it is. As he explained:

> "If the point t is very far from the initial point t_0, and if β is positive so that there is actually a damping, then the influence of the initial situation y_0 and $\dot{y}_0$ on the ordinate $y(t)$ will be negligible, that is, the ordinate will be
>
> $$y\left(t\right) = \sum_{k=1}^{n} Q\left(t - t_k\right) e_k \quad (41)[\text{eq. 8, p. 200}]$$
>
> This means that the ordinate $y(t)$ of the pendulum at a given moment will simply be the *cumulation* of the effects of the shocks, the *cumulation* being made according to a system of weights" (1933, p. 201).

With Eq. (41), Frisch established his PIPP model **without explicitly saying so**. Then he clarified that "these weights are simply the shape of the function $Q(\tau)$. That is to say, $y(t)$ is the result of applying a linear operator to the shocks, and *the system of weights in the operator will simply be given by the* *shape of the time curve that would have been the solution of the determinate dynamic system in case the movement had been allowed to go on undisturbed.* The fundamental question which arises is, therefore: If we perform a cumulation where the weights have the form $Q(\tau)$, what sort of time shape will the function $y(t)$ get?" (1933, pp. 201-202)

We understand that Frisch's concept of impulses is based on a discretization of time. He considered that each time the damped pendulum reaches its maximal amplitude positions, i.e., at the points of time $t_1, t_2, \dots t_n$, it receives "shocks which may be directed either in the positive or in the negative sense and that may have arbitrary strengths." Then, by neglecting the influence of initial conditions, Frisch (1933, 201) stated that the motion of the damped oscillations maintained by erratic shocks can be represented by Eq. (41) [eq. 8, p. 201]. This idea of Frisch corresponds precisely to our metaphor of the swing for children.

4.2 Frisch's PPIP model: an incomplete model

While Frisch can exhibit his PPIP model (i.e., Eq. 41), unfortunately, his model has three major drawbacks that make it incomplete.

First drawback, Frisch did not specify how many shocks are necessary to represent as accurately as possible the economic oscillations. In other words, he did not provide any value of the upper bound of n in the Eq. (41) [eq. 8, p. 201] which depends, of course, on the shape and length of the time series considered. Nevertheless, Frisch (1933, 202) did not give more details about his concept of impulses, but he "only" explained that irregular economic fluctuations can be modeled with damped oscillations maintained by erratic shocks.

Second drawback, Frisch did not specify the strength of each shock e_k which should be applied to maintain these damped oscillations.

Third drawback, Frisch did not indicate how to estimate, from the time series of statistical data, its *damped period* or its *damped frequency* α and the *damping exponent* or *damping factor* β. In other words, this estimation of parameters, which is deduced from a specific and given time series, prevent from any universal model[15].

Given these drawbacks, Frisch's PPIP model is necessarily incomplete. In other terms, such model cannot be generalized; the solution will always be empirical.

5 A reconstruction of Frisch's propagation-impulse model

The previous section has established that Frisch's PPIP model is necessarily incomplete. This is a fundamental result. Given that we cannot have a com-

[15] Let's notice that although the perturbations produced by such erratic shocks make vary the damped period and the amplitude, such variations, taking place "within such limits that it is reasonable to speak of an *average* period and an *average* amplitude", do not correspond to the *moving average* as it can be easily evidenced.

plete mathematical formulation of Frisch's MODEL 3, we cannot test it with a methodology implying hypotheses and then statistical tests. For this reason, Frisch had to calibrate his model thanks to an empirical estimation of the parameters. The calibration leads to the identification of unconditional parameters of the mathematical model, which can be then tested. If Frisch could calibrate his model, he could not investigate for determining unconditional parameters, which would necessitate the use of computers given the complexity of the PPIP model for providing any extensive numerical computations or any mathematical formulation of his model.

In his essay, Frisch only provided a figure page 202 (i.e., figure 6), but he did not provide any proof, claiming that

> "I shall not attempt to give any formal proof of these facts here. A detailed proof, together with extensive numerical computations, will be given in the above-mentioned paper to appear in *Econometrica*" (1933, p. 202).

Because the later paper was never published, to date we did not have any proof that this figure can really be obtained by calibrating the PPIP model. In this section, we reconstruct Frisch's figure 6 and validate his intuition that *free damped oscillations* can be maintained by erratic shocks. For doing this reconstruction, we will follow two steps; in the first step, we will reproduce Frisch's data; in the second step, we will see if we can calibrate the model as Frisch was supposed to have done (i.e., estimate the value of the parameters used by Frisch).

First step, we will reproduce Frisch's data. As Frisch explained, with his PPIP model, his goal was "to reproducing the graph [see Fig. 7 reproduced below] of a changing harmonic produced experimentally as the cumulation of erratic impulses, the weight function being of the form [41]" (1933, p. 202). Frisch plotted the irregular oscillations of statistical data and the graph of his PPIP model, which is the sine damped function represented by Eq. (41) [eq. 8, p. 201] in which $Q(\tau)$ is given by Eq. (38) [eq. 4, p. 200]:

$$y\left(t\right)=\sum_{k=1}^{n}Q\left(t-t_k\right)e_k \text{ where } Q\left(\tau\right)=\frac{1}{\alpha}e^{-\beta\tau}\sin\left(\alpha\tau\right) \tag{42}$$

or,

$$y\left(t\right)=\frac{1}{\alpha}\sum_{k=1}^{n}e^{-\beta\left(t-t_k\right)}\sin\left[\alpha\left(t-t_k\right)\right]e_k \tag{43}$$

However, Frisch did not provide any demonstration for supporting his Fig. 7. He didn't specified neither the shock number k nor the strength of each shock e_k necessary to maintain the cycles he presented on his figure 6 (here Fig. 7). In order to reproduce Frisch's approach as accurately as possible, we have extracted the statistical data he used in Fig. 7 with the help of signal processing methods which allows us to converter scanned graphs to data (i.e.,

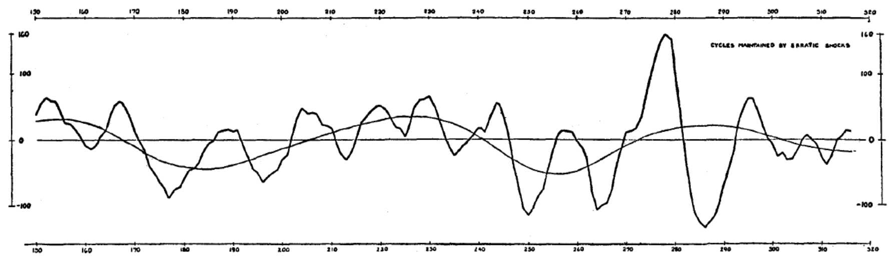

Fig. 7 Cycles maintained by erratic shocks, from Frisch (1933, p. 202).

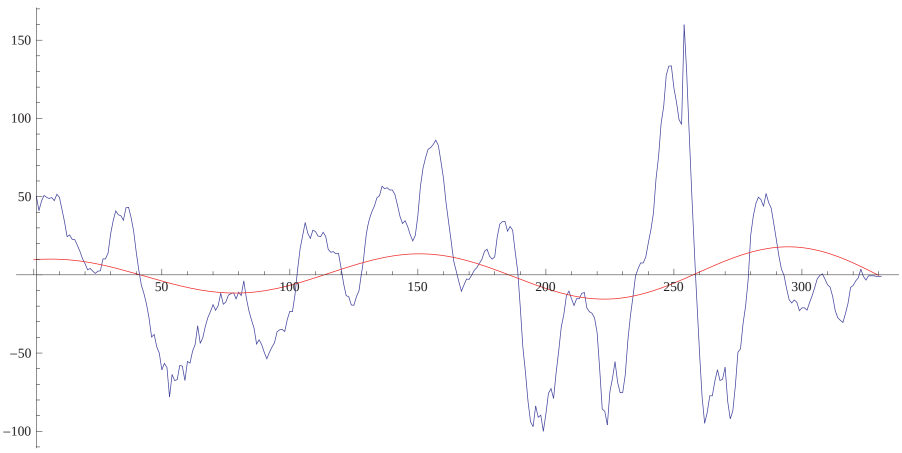

Fig. 8 Statistical data in blue and Eq. (43) in red..

we have extracted (x,y) data from scanned graphs can be useful for analyzing data from published graphs). These data are represented in blue in Fig. 8.

Using classical methods, we have graphically deduced from Fig. 7 [Fig. 6, p. 202], and then reproduced in Fig. 8, the *damped period* and so, the *damped frequency* α of Frisch's function $Q(\tau)$ (see Eq. (38) above). As it can be read directly on Fig. 7, $T/2 = 72$. So, the *damped period* $T = 144$ and the *damped frequency* $\alpha = 1/T = 0.00694$. By using the *logarithmic decrement* δ, which is defined as the natural logarithm of the ratio of the amplitudes of any two successive peaks, i.e., any two successive maxima of amplitude, we found that $\delta = \ln(30/40) = -0.28$. Since we have the well-known relationship $\delta = \beta T = \beta/\alpha$, we obtained that the value of the *damping exponent* or *damping factor* $\beta = -0.002$.

The next step is to calibrate the PPIP model as Frisch did with his own data and therefore check if it is possible to reproduce with the function (43) the graph that Frisch plotted in his figure 6 (see Fig. 7 and 8). In so doing, we can properly reproduce how Frisch arrived at his results. First of all, it is worth clarifying that we had to overcome the three drawbacks highlighted in the previous section. If we could graphically find the *damping frequency* α and the *damping exponent* β, the question of the number n of erratic shocks to take into account is still unanswered. Moreover, the same applies for the strength (values) of the erratic shocks e_k either positive or negative which seem to be obtained empirically. However, surprisingly as it may seem, we have been able to approximately reproduce Frisch's figure 6 (see Fig. 7 and 8) by using the PPIP model (eq. 43) with only $n = 4$ erratic shocks the strength of which has been chosen *ad hoc*[16].

The striking similarity between Frisch's figure 6 (see Fig. 7) and Fig. 8 enables to prove that it is indeed possible to maintain cycles by erratic shocks. In other words, damped oscillations can be maintained by erratic shocks as it is exactly the case for the children's swing. Moreover, our work proves that Frisch may have calibrated his model for testing it with calibration technic. In fact, with this simple application of his concept of impulses, we show that the weighted sum of damped sine functions can give rise to maintained oscillations in Frisch's PPIP model. As a consequence of this fundamental result, it can be easily proved that the weighted sum of Frisch's damped sine functions (see Eqs. (19) [18, p. 190], Sec. 3.2.4), solution of his system (11), can also give rise to maintained oscillations thanks to erratic shocks.

6 Conclusion

The contribution of Frisch's 1933 book chapter to econometrics and macroeconomics has been clearly acknowledged by economists. However, the understanding of Frisch's 1933 essay and its contribution are less obvious than previous studies have suggested. By providing the first comprehensive mathematical analysis of Frisch's 1933 essay, we have proven that 1) Frisch's MODEL 2 rocks; 2) Frisch's PPIP model is mathematically incomplete; 3) and the latter could have been calibrated by Frisch.

Investigating Frisch's essay through the mathematical perspective has given the opportunity to show that Frisch's analysis and demonstration are based on the new methods of the celestial mechanics developed by Poincaré. So doing, we can see that the contribution of Frisch's book chapter goes far beyond what has been considered so far. With his essay, Frisch introduced a new kind of oscillation, the *damped oscillations* maintained by erratic shocks. Such a oscillations have been extended by French and Russian mathematicians. Frisch's new oscillations renamed *impulse-excited oscillations* Andronov and Khaikin (1937) in their "Theory of the clock", and then popularized by Minorsky (1947). The applications of this kind of oscillations are nowadays investigated in applied mathematics and physics. The present article shows that the place of Frisch's 1933 essay in the development of the theory of oscillations deserves to be investigated more deeply.

[16] Of course, the increase of the number n of erratic shocks would improve the accuracy of the modeling.

Acknowledgements Author would like to thank the reviewers who considerably improved this work with their remarks and helpful advices.

Conflict of interest

The author declares that he has no conflict of interest.

References

1. Akerman, J. (1928). Om det Ekonomiska Livets Rytmik [Rhythmics of Economic Life] University of Lund]. Lund.
2. Andronov, A. A. & Khaikin, S. E. (1937 [1949]). Theory of oscillations (Transl. and adapted by S. Lefschetz). Princeton University Press.
3. Arrow, K. J. (1960). The Work of Ragnar Frisch, Econometrician. Econometrica, 28(2), 175-192.
4. Backhouse, R. E. (2015). Revisiting Samuelson's "Foundations of Economic Analysis". Journal of Economic Literature 53(2), 326-350.
5. Bjerkholt, O. (2007). Ragnar Frisch's contribution to business cycle analysis. European Journal of the History of Economic Thought, 13(3), 449-486.
6. Bjerkholt, O. & Dupont, A. (2010). Ragnar Frisch's Conception of Econometrics. History of Political Economy, 42(1), 21-73.
7. Blatt, J. M. (1980). On the Frisch Model of Business Cycles. Oxford Economic Papers 32(3), 467-479.
8. Boumans, M. (2005). How Economists Model the World into Numbers, Routeledge, London & New York, Taylor & Francis.
9. Boumans, M. (2005). Frisch on testing of business cycle theories. Journal of Econometrics, 67(1), 129-147.
10. Carret, V. (2020). Ragnar Frisch 1933 model: And yet it rocks! HAL archives-ouverts.fr.
11. Christ, C. F. (1983). The Founding of the Econometric Society and Econometrica. Econometrica, 51(1), 3-6.
12. Christiano, L. J., Eichenbaum, M. S., & Trabandt, M. (2018). On DSGE Models. Journal of Economic Perspectives, 32(3), 113-140.
13. Cooke, K. L. (1963). Differential - Difference Equations. In International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics, Editor(s): Joseph P. LaSalle, Solomon Lefschetz, Academic Press, 1963, 155-171,
14. Dupont-Kieffer, A. (2012). The accelerator principle at the core of Frisch's 1933 rocking horse model. Journal of the History of Economic Thought, 34(4), 447-473.
15. Duque, R. (2009). The Rocking Horse Reloaded: An overview over Ragnar Frisch's 1933 Propagation and Impulse Problems University of Oslo].
16. Frisch, R. (1931). The Interrelation Between Capital Production and Consumer-Taking. Journal of Political Economy, 39(5), 646-654.
17. Frisch, R. (1933). Propagation problems and impulse problems in dynamic economics. In Economic essays in honour of Gustav Cassel (pp. 171-205). George Allen and Unwin Ltd.
18. Frisch, R. & Holme, H. (1935). The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics. Econometrica, 3(2), 225-239.
19. Fröberg, C.-E. (1985). Numerical Mathematics: Theory And Computer Applications. Basic Books.
20. Ginoux, J.-M. (2017). History of nonlinear Oscillations Theory in France (1880-1940). Springer International Publishing.
21. Ginoux, J.-M., Jovanovic, F., Meucci, R. & Llibre, J. (2022). Rocard's 1941 chaotic relaxation econometric oscillator, International Journal of Bifurcation and Chaos, March 2022, forthcoming paper.
22. Hamburger, L. (1931). Analogie des fluctuations économiques et des oscillations de relaxation. Indices du Mouvement des Affaires, 9 - supplément(Janvier), 1-35.

23. Hansen, L. P. & Heckman, J. J. (1996). The Empirical Foundations of Calibration. The journal of economic perspectives, 10(1), 87-104.
24. Hotelling, H. (1927). Differential Equations Subject to Error, and Population Estimates. Journal of the American Statistical Association, 22(159), 283-314.
25. Jovanovic, F. & Ginoux, J.-M. (2020). The 'Rocking Horse Model Does Rock': Solving Zambelli's Puzzle (October 17, 2020). Available at SSRN: https://ssrn.com/abstract=3713959 or http://dx.doi.org/10.2139/ssrn.3713959
26. Juglar, C. (1889). Des crises commerciales et de leur retour periodique en France, en Angleterre et aux Etats-Unis (2nd ed.). Guillaumin.
27. Kalecki, M. (1935). A Macrodynamic Theory of Business Cycles. Econometrica, 3(3), 327-344.
28. Le Gall, P. (1994). Histoire de l'Econométrie, 1914-1944. L'Erosion du Déterminisme [Doctoral dissertation, Université Paris I Panthéon-Sorbonne].
29. Lord Rayleigh (John William Strutt). (1883). On maintained vibrations. Philosophical Magazine, Series 5, 15(94), 229-235.
30. Louçà, F. (2007). The years of high econometrics. A short history of the generation that reinvented economics. Routledge.
31. Minorsky, N. (1942). Self-excited Oscillations in Dynamical Systems Possessing Retarded Actions. Journal of Applied Mechanics, 9(2), A65-A71.
32. Minorsky, N. (1947). Introduction to non-linear mechanics (Originally published 1944-1946, as Restricted reports by David W. Taylor Model Basin, U.S. Navy). J. W. Edwards.
33. Minorsky, N. (1948). Self-excited mechanical oscillations. Journal of Applied Physics 19, 332-338.
34. Mitchell, W. C. (1913). Business Cycles. University of California Press.
35. Morgan, M. S. (1990). The history of econometric ideas. Cambridge University Press.
36. Poincaré, H. (1992 [1892, 1893, 1899]). New Methods of Celestial Mechanics, edited by Daniel Goroff (3 volumes). English translation of the French original edition of Les Méthodes Nouvelles de la Mécanique Céleste, Paris, Gauthier-Villars, 1892, 1893, 1899 (3 volumes). American Institute of Physics.
37. Rocard, Y. (1941). Théorie des oscillateurs, Édition de la Revue Scientifique.
38. Rudin, W. (1966). Real and Complex Analysis. McGraw-Hill.
39. Samuelson, P. A. (1974). Remembrances of Frisch. European Economic Review, 5(1), 7-23.
40. Slutsky, E. (1927). Slozhenie sluchainykh prichin, kak istochnik tsiklicheskikh protsessov. Voprosy kon"yunktury, 3, 34-64.
41. Thalberg, B. (1990). A Reconsideration of Frisch's Original Cycle Model. In K. Velupillai (Ed.), Nonlinear and Multisectoral Macrodynamics. Essays in Honour of Richard Goodwin (pp. 96-117). The MacMillan Press Ltd.
42. Tinbergen, J. (1935). Annual Survey: Suggestions on a Quantitative Business Cycle Theory. Econometrica, 3(3), 241-308.
43. Van der Pol, B. (1925). Het onderling verband tusschen eenige moderne vorderingen in de draadlooze telegrafie en telefonie,("The Interrelation of Some Modern Advances in Wireless Telegraphy and Telephony"). Polytechnish Weekblad, 19, 791-794.
44. Van der Pol, B. (1926). On relaxation-oscillations. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 2(VII), 978-992.
45. Velupillai, K. (1992). Implicit Nonlinearities in the Economic Dynamics of 'Impulse and Propagation'. In K. Velupillai (Ed.), Nonlinearities, Disequilibria and Simulation - Proceedings of the Arne Ryde symposium on Quantitative Methods in the Stabilization of Macrodynamic Systems. Essays in honor of Bjorn Thalberg. The MacMillan Press Ltd.
46. Venkatachalam, R., & Velupillai, K. (2012). Origins and Early Development of the Nonlinear Endogenous Mathematical Theory of the Business Cycle. Economia Politica, 24(1), 45-79.
47. Yule, G. U. (1927). On a Method of Investigating Periodicities in Disturbed Series, with Special Reference to Wolfer's Sunspot Numbers. Philosophical Transactions of the Royal Society of London, Series A, 226, 267-298.
48. Zambelli, S. (1992). The Wooden Horse that Wouldn't Rock: Reconsidering Frisch. In K. Velupillai (Ed.), Nonlinearities, Disequilibria and Simulation - Proceedings of the Arne Ryde symposium on Quantitative Methods in the Stabilization of Macrodynamic Systems. Essays in honor of Bjorn Thalberg. The MacMillan Press Ltd.