The practical use of the general formula of the optimal hedge ratio in option pricing: an example of the generalized model of an exponentially truncated Levy stable distribution

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Abstract

In financial option pricing, the optimal hedge ratio is a well known concept. This paper uses this concept within the context of an exponentially truncated Lévy stable distribution.

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1 Introduction

This paper deals with the possibility to price options by using a specific statistical processes called truncated stable Lévy process. More precisely, we provide a generalized model inspired from the Black and Scholes framework in which an admissible hedging strategy is possible in a sense defined by Harrison and Kreps [1]. In the 1990s, the growing evidence of power law properties in financial data generated renewed interest in stable Lévy processes which re-emerged through three categories [1] of models which can be identified depending on the technical solution developed to escape from the infinite variance problem encountered by economists in the 1970s: time-changed processes, models using a specific calibration of parameters and models based on a particular calibration (truncation) of a stable distribution. The *first* category refers to time-changed models which introduce an "intrinsic time" providing tail effects as observed in the market (see for instance Hurst et al. [6]). In other words, the leptokurticity of empirical distributions is generated by a stable Lévy process-time subordinated to the Wiener process. The *second* category of models refers to a specific calibration of parameters in order to have finite statistical moments. Empirical distributions or time are not changed but authors give a specific statistical conditions implying the existence of all statistical moments (see for instance Cartea and Howison [7]). The *third* category of statistical solutions developed to solve the problem of infinite variance includes all models based on a particular calibration of distribution. The difference between this category of models and those of the previous category described above refers to the kind of calibration used. See for instance Geman, Madan and Yor [8].

In this article, we propose a general model to price options when the stochastic price process is not characterized by a Gaussian or log-normal distribution as is the case in the Black and Scholes model [9]. There are many authors working on this subject due to the failure of the Gaussian distribution to fit the empirical distribution, but in general these works deal with complicated stochastic calculus and there is not yet any algorithm which is generally accepted for the evaluation of options with non-Gaussian models. Moreover, except in specific Brownian models, risk in option trading cannot be eliminated. In this case, the minimization of a particular risk measure is required. That will be the case in this paper where we will use the risk measure defined by Bouchaud and Sornette [10] and Aurell et al. [11] but we differ from this article by presenting a more generalized derivation directly in line with the mathematical framework developed by Harrison and Kreps [1]. More precisely, we will show that our model meets the sufficient condition of arbitrage defined by Harrison

[1] In these categories, we do not include the jump-diffusion/pure jump models neither the ARCH models which are non stable processes. See Eberlein and Keller [2]; Carr, Geman, Madan and Yor [3]; Borland [4]; Cont and Tankov [5].

and Kreps [1].

The first section of this paper will present a generalized model in line with the Black and Scholes framework, while the second section will define a specific risk measure for a non-Gaussian version of this model that we then will present in the third section. We will use an exponentially truncated Lévy distribution whose statistical conditions will be defined to make this model viable in a sense defined by Harrison and Kreps [1].

2 The model

Let us consider a portfolio made up of a (call) option and a short position on stocks. At time $t = 0$, the value of this portfolio V is: $V = C - \phi S$, where C is the price of the call option (with strike price K and expiration T); S is the stock price (vector) and ϕ is the quantity of the stock. The product $\phi S = \sum_i \phi_i S_i$ is to be seen as a scalar product with S_i the price for each stock. Initially the stock price is considered to be S_0 and the portfolio is considered to be self-financing. In other words the value of the portfolio changes only due to changes in the stock price. In this situation the variation of the portfolio between time $t = 0$ and T is given by: $\Delta V = \Delta C - \phi \Delta S$.

The variation of the portfolio due to the call option is, when continuously discounted at the risk free rate r:

$$\begin{aligned}\exp(-rT)\Delta V = & \exp(-rT)\max(S-K,0) - \exp(r0)C(S_0,K,T) \\ & - \exp(-rT)\phi\Delta S,\end{aligned} \tag{1}$$

where the first term is the value of the option at time T and the second term is the premium paid for the option at time $t = 0$.

Assuming, we operate in a no-arbitrage market then the discounted stochastic price process must be a martingale (Harrison and Kreps [1]). We get for the option price:

$$C(S_0,K,T) = e^{-rT}E[\max(S-K,0)] = e^{-rT}\int_K^\infty (S-K)f(S)dS, \tag{2}$$

where $f(S)$ is the probability measure that makes the price to be a martingale. Thus we cannot use the above equation with any probability distribution function in order to price options. It is only valid with distributions providing at least a martingale measure (Harrison and Kreps [1], p.383). The Black and Scholes solution for option pricing is obtained from the above equation by using for $f(S)$ the log-normal distribution. In this article, we will use this model by substituting the log-normal distribution by an exponentially truncated Lévy one. Since we will work with a non-Gaussian framework, we will have to use a specific risk measure which is defined in the following section.

3 Evaluation of risk

Due to the stochastic nature of the price process, risk is inherent to the financial evaluation of options and stocks. For the log-normal distribution it was shown by Black and Scholes [9] that this risk can be completely eliminated (or hedged) by using an appropriate hedging condition (the so-called ϕ hedging) for the financing strategy. But for non-normal models, the Black and Scholes procedure for hedging risk does not work anymore. A measure of risk that was used in Bouchaud and Sornette [10] and Aurell et al. [11] is the variance of the value of the portfolio $V = C - \phi S$. We make the supposition here that this variance is finite. Thus:

$$R = E\left[\Delta V^2\right] = E[(\max(S-K,0) - C(S_0,K,T) - \phi\Delta S)^2]. \tag{3}$$

First of all let us note that for uncorrelated assets, we have the following expression: $E\left[(\phi\Delta S)^2\right] = \sum \phi_i^2\sigma_i^2$, where σ_i is the volatility. However, when there exists a correlation between the assets, we can write: $E\left[(\phi\Delta S)^2\right] = \sum \phi_i^2\sigma_i^2 + 2\sum_{i,j}\phi_i\phi_j\sigma_{ij}$, where σ_{ij} is the covariance matrix. In a first time, let us consider a simplified situation with only one stock, of volatility σ. In this case evaluating equation (3) and minimizing the risk with respect to the trading strategy we get an optimal trading strategy that minimizes the risk:

$$\phi^* = \frac{1}{\sigma^2}E\left[(S_0 - S)\max(S-K,0)\right] = \frac{1}{\sigma^2}\int_K^\infty (S_0 - S)(S-K)f(S)dS. \tag{4}$$

This equation is valid for a martingale process S with $E[\Delta S] = 0$. If there are more than one uncorrelated assets (stocks) the above equation should be applied for each stock individually in order to get the total optimal hedging strategy. The optimal strategy for the i'th asset would be written like the above equation with index i on all variables (except thus K). For many correlated assets, using $E\left[(\phi\Delta S)^2\right] = \sum \phi_i^2\sigma_i^2 + 2\sum_{i,j}\phi_i\phi_j\sigma_{ij}$, one finds:

$$\phi_i^* = \frac{1}{\sigma_i^2}\left[\int_K^{\infty}(S_{i0}-S_i)(S_i-K)f(S_i)dS_i - \sum_j \phi_j\sigma_{ij}\right] \tag{5}$$

In the simplest case it is straightforward to observe that in the case of the normal distribution with log-returns the optimal hedging strategy given in equation (4) is the same as the hedging strategy from the Black and Scholes model, i.e. $\phi^* = \frac{dC}{dS}$. The minimal risk R corresponding to the optimal hedging strategy is obtained from equation (3): $R^* = R_C - \phi^{*2}\sigma^2$ for one stock. Note that R_C is a risk term not-dependent upon the investment strategy, defined as: $R_C = \int_K^{\infty}(S-K)^2 f(S)dS - \left(\int_K^{\infty}(S-K)^2 f(S)dS\right)^2$. In the general case with many correlated assets, the minimal risk is obtained by taking into account $E\left[(\phi\Delta S)^2\right] = \sum \phi_i^2\sigma_i^2 + 2\sum_{i,j}\phi_i\phi_j\sigma_{ij}$. Bouchaud and Sornette [10] show that R^* vanishes when a log-normal density is used.

4 Option pricing with exponentially truncated Lévy stable distribution and the finding of the hedging strategy for this model

In this section, we use the generalized model presented in the first section associated with an exponentially truncated Lévy stable distribution. This approach will require the use of the risk measure defined in the previous section. Let us consider the distribution density for the log returns defined by the equation: $f(x) = \overline{C}\frac{e^{-\gamma|x|}}{|x|^{\alpha+1}}$, where $x = \log\left(\frac{S}{S_0}\right)$, $\overline{C} > 0$, $\gamma \geq 0$ and $0 < \alpha < 1$ which is the condition to have a stable Lévy distribution. See Bucsa et al. [12]. $\overline{C}$ can be seen as a measure of the level of activity while the parameter γ is the speed at which arrival rates decline with the size of the move (i.e rate of exponential decay). This model is a symmetric version of the so-called CGMY model (Carr et al. [3]) and a generalization of the exponentially truncated Lévy stable model (Koponen [13]) in the limit of high return values. Although we use a stable Lévy process, the exponentially truncation implies a exponential decay of the distribution. This restriction means that the truncated distribution generates finite variations making possible the estimation of the variance which is given by the following equation:

$$\sigma^2 = 2\overline{C}\gamma^{\alpha-2}\Gamma(2-\alpha) \text{ with } \Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt. \tag{6}$$

Using equation (2), we calculate the option price for this model for the chosen portfolio, by considering the density distribution of stock returns:

$$C = e^{-rT} \int_{\ln\left(\frac{K}{S_0}\right)}^{\infty} \left(S_0 e^{-x} - K\right) \overline{C} \frac{e^{-\gamma x}}{x^{\alpha+1}} dx. \tag{7}$$

Using the result: $\int_x^\infty \frac{e^{-u}}{u^n} du = \frac{E_n(x)}{x^{n-1}}$ with $E_n(x)$ the general exponential integral. Using this result in equation (7), and expressing $\overline{C}$ as a function of squared volatility, yields:

$$C = \frac{\sigma^2 e^{-rT}}{2\gamma^{\alpha-2}\Gamma(2-\alpha)} \left[\ln\left(\frac{K}{S_0}\right)\right]^{-\alpha} \left\{ \begin{array}{c} S_0 E_{\alpha+1}\left[(\gamma-1)\ln\left(\frac{K}{S_0}\right)\right] - \\ K E_{\alpha+1}\left[\gamma \ln\left(\frac{K}{S_0}\right)\right] \end{array} \right\}. \tag{8}$$

Given this result, we can estimate the hedging strategy minimizing the risk by using equation (4):

$$\begin{aligned} \phi^* = \frac{\overline{C}}{\sigma^2} \int_{\ln\left(\frac{K}{S_0}\right)}^{\infty} \left(S_0 - S_0 e^x\right)\left(S_0 e^x - K\right) \frac{e^{-\gamma x}}{x^{\alpha+1}} dx = \\ \frac{1}{2\gamma^{\alpha-2}\Gamma(2-\alpha)} \left[\ln\left(\frac{K}{S_0}\right)\right] \times \\ \left\{ \begin{array}{c} \left(S_0 K + S_0^2\right) E_{\alpha+1}\left[(\gamma-1)\ln\left(\frac{K}{S_0}\right)\right] - \\ S_0^2 E_{\alpha+1}\left[(\gamma-2)\ln\left(\frac{K}{S_0}\right)\right] - \\ S_0 K E_{\alpha+1}\left[\gamma \ln\left(\frac{K}{S_0}\right)\right] \end{array} \right\}. \end{aligned} \tag{9}$$

5 Restrictions for arbitrage opportunities and concluding remark

The result ϕ^* which is proposed in equation (9) above can also be generalized. Tan [15] shows that for non-Gaussian densities, ϕ^* will explicitly depend on i) higher partial derivatives of the call option pricing function towards the price of the underlying asset; and ii) the value of the cumulants (as they are used in the logarithm of the characteristic function). We do not expand on this result in this paper. The use of a symmetric distribution allows us to have the sufficient condition to have, at least, a martingale measure. Although the uniqueness of this martingale is not obvious for stable Lévy processes, the exponential truncation combined with the symmetry condition ensures the condition for the existence of at least one martingale measure. In this perspective, our model is line with Harrison and Kreps' [1] sufficient condition but not with the necessary condition to be in a no-arbitrage and complete market (Harrison and Pliska [14]).

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